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Kicking the Rugby Ball: Perturbations of 6D Gauged Chiral Supergravity

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Abstract: We analyze the axially-symmetric scalar perturbations of 6D chiral gauged supergravity compactified on the general warped geometries in the presence of two source branes. We find all of the conical geometries are marginally stable for normalizable perturbations (in disagreement with some recent calculations) and the nonconical for regular perturbations, even though none of them are supersymmetric (apart from the trivial Salam-Sezgin solution, for which there are no source branes). The marginal direction is the one whose presence is required by the classical scaling property of the field equations, and all other modes have positive squared mass. In the special case of the conical solutions, including (but not restricted to) the unwarped 'rugby-ball' solutions, we find closed-form expressions for the mode functions in terms of Legendre and Hypergeometric functions. In so doing we show how to match the asymptotic near-brane form for the solution to the physics of the source branes, and thereby how to physically interpret perturbations which can be singular at the brane positions.

Contents

1. Introduction

Six dimensional supergravity continues to provide an instructive laboratory for exploring different aspects of the cosmological constant problem [1, 2]. It does so by building on the earlier observation [3, 4, 5, 6] that higher-dimensional brane constructions can break the implication that says that a nonzero 4D energy distribution must necessarily generate a large 4D curvature. Once recast into a higher-dimensional context the usual cosmological constant problems [7] become questions as to what are the choices required for obtaining acceptably flat 4D space, and identifying how natural these choices may be. Part of the appeal of the 6D proposal is that it is very predictive, having observational implications both for accelerator physics and for tests of gravity in the lab and in astrophysics [8], in addition to its potential implications for late-time cosmology [9].

In extra-dimensional models there are two separate kinds of naturalness issues which must be addressed. The first type of naturalness demands stability under renormalization for any choices which are required to ensure the flatness of 4D space. That is, any choice of brane positions or couplings which are required for 4D flatness must be stable against integrating out the ultraviolet part of the theory. Although extra-dimensional models appear to make progress in understanding this kind of stability [10], it remains open whether they can completely resolve this part of the problem.

The second type of naturalness issue arises in extra-dimensional models because these models generically relate the Dark Energy density to the dynamics of the various moduli fields which describe the geometry of the internal dimensions. Although it is possible to obtain phenomenologically successful cosmology from this kind of dynamics [9], the second naturalness issue asks whether this requires unusually delicate choices for the brane and bulk initial conditions. In particular one worries that generic initial conditions — or small perturbations to specially-chosen ones — might give rise to catastrophically fast time evolution, leading to late-time cosmology which is unlike what we see around us.

The present paper addresses the second of these naturalness issues. Any understanding of this issue requires knowledge of the time-dependent solutions of the system. And since the time scales of interest are not slow compared to those associated with the Kaluza Klein states, this time evolution is intrinsically higher-dimensional and cannot be accurately described within an effective 4D description [11]. Instead, what is required is an intrinsically higher-dimensional description of the system's dynamics and its response to various probes.

In a companion paper [12] we present a broad class of time-dependent scaling solutions for 4D compactifications of 6D supergravity. These solutions aid in understanding the evolution produced by generic brane properties by showing what the late-time evolution of the system is likely to be for a wide class of initial conditions. The present paper is aimed at a complementary part of the problem of time-dependence. Instead of seeking the late-time behaviour towards which the system evolves, we here study how small perturbations can initiate the beginnings of time-dependent behaviour. (See ref. [13] for a description of some of the intermediate and transient time-dependent phenomena which lies between these two cases.) We do so by investigating the linear perturbations of a broad class of static solutions to 6D supergravity, in order to see how they respond to arbitrary small perturbations. In this analysis, we use a Kaluza Klein decomposition which includes both the modes oscillating in time as well as the zero-mass non-oscillating ones.

Our main result is that these systems are marginally stable for normalizable perturbations around conical geometries, and for regular perturbations around nonconical ones. (We also examine a class of physically motivated non-regular perturbations, for which our stability analysis is inconclusive.) The stability is only marginal because of the generic scaling property which the 6D supergravity solutions have. (It is the absence of an energy barrier in this marginal direction which allows the development of the scaling solutions studied in ref. [12].) We identify the set of equations which govern the time evolution of the system after all of the constraints have been used to eliminate some of the field fluctuations, and reproduce the form found by earlier workers [14]. But we differ in the conclusions which we draw regarding stability from these equations, for reasons we explain in detail in the Appendix.

We begin our discussion in §2 with a reminder of the field equations and some of the known static background configurations. This is followed in §3 by the derivation of the equations governing linearized perturbations in comoving gauge, while the same derivation in longitudinal gauge is performed in appendix A for comparison with previous results. §4 then solves analytically the linearized equations in the conical case and identifies the asymptotic behaviour in the more general non-conical case. We then give in §5 several general arguments in favour of the stability of the modes examined and relate them with the boundary conditions in §4. After a brief summary of conclusions – §6 – four appendices contain: details of the analysis in longitudinal gauge, with the full stability argument; a detailed comparison of why our stability conclusions differ from some previous results; the relation between some of the results in various gauges; and a summary of some properties of the special functions used in the explicit solutions.

2. Field Equations and Background Solutions

We begin in this section by summarizing the relevant field equations, as well as the static background solutions about which we shall perturb.

2.1 Field equations

The bosonic part of the Lagrangian density for 6D chiral gauged supergravity is given (for the case of vanishing hyperscalars — $\Phi^a = 0$) by [15][1]

$$\frac{\mathcal{L}}{\sqrt{-g}} = -\frac{1}{2\kappa^2}\, g^{MN}\Big[R_{MN} + \partial_M\varphi\,\partial_N\varphi\Big] - \frac{2g^2}{\kappa^4}\, e^{\varphi} \\ -\frac{1}{4}\, e^{-\varphi}\, F_{MN}F^{MN} - \frac{1}{2\cdot 3!}\, e^{-2\varphi}\, G_{MNP}G^{MNP}\,, \tag{2.1}$$

[1]The curvature conventions used here are those of Weinberg's book [16], for which all curvature tensors differ by an overall sign relative to those of MTW [17].

where $F_{MN} = \partial_M A_N - \partial_N A_M$ and $G_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN} + (A_P F_{MN}\text{ terms})$. The coupling constants g and κ respectively have dimension $(\text{mass})^{-1}$ and $(\text{mass})^{-2}$.

The field equations obtained from this action are:

$$
\begin{aligned}
\Box\,\varphi + \frac{\kappa^2}{6}\, e^{-2\varphi}\, G_{MNP}G^{MNP} + \frac{\kappa^2}{4}\, e^{-\varphi}\, F_{MN}F^{MN} - \frac{2\,g^2}{\kappa^2}\, e^{\varphi} &= 0 \qquad &\text{(dilaton)}\\
D_M\Big(e^{-2\varphi}\, G^{MNP}\Big) &= 0 \qquad &\text{(2-Form)}\\
D_M\Big(e^{-\varphi}\, F^{MN}\Big) + e^{-2\varphi}\, G^{MNP} F_{MP} &= 0 \qquad &\text{(Maxwell)}\\
R_{MN} + \partial_M\varphi\,\partial_N\varphi + \frac{\kappa^2}{2}\, e^{-2\varphi}\, G_{MPQ}G_N{}^{PQ} + \kappa^2 e^{-\varphi}\, F_{MP}F_N{}^P + \frac{1}{2}\,(\Box\,\varphi)\, g_{MN} &= 0\,. \qquad &\text{(Einstein)}
\end{aligned}
\tag{2.2}
$$

An important feature of these equations is their classical scaling property. This property states that given any solution to these equations another can be obtained by making the replacement

$$
e^{\varphi} \to \lambda e^{\varphi} \qquad \text{and} \qquad g_{MN} \to \lambda^{-1} g_{MN}\,, \tag{2.3}
$$

with all other fields held fixed. This property follows from the fact that the action, eq. (2.1), scales under this transformation according to $S \to \lambda^{-2} S$ [18].

The remainder of the paper deals with solutions to these equations for which two dimensions are compact and four are not. To this end denote the six coordinates x^M, $M = 0,1,2,3,4,5$, as $x^M = \{t,x,y,z,\eta,\theta\}$. We denote 4D coordinates by x^μ, $\mu = 0,1,2,3$, where $x^\mu = \{t,x,y,z\}$, and 2D coordinates of the extra dimensions by x^m, $m = 4,5$, where $x^m = \{\eta,\theta\}$. When required, the three spatial coordinates of the observable four dimensions are denoted $x^i, i = 1,2,3$ and so $x^i = \{x,y,z\}$.

2.2 Ansätze

In later sections our interest is in metrics of the form

$$
\mathrm{d}s^2 = e^{2a}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2v}\,\mathrm{d}\eta^2 + e^{2b}\mathrm{d}\theta^2 \tag{2.4}
$$

and consider these component functions, as well as φ and A_θ, to depend only on the coordinates η and t. (In what follows we sometimes generalize this assumption to allow dependence on η and x^μ.) Denoting differentiation with respect to η and t by primes and dots, the field equations for these functions then reduce to the following set of coupled partial differential equations.

The Maxwell equation is:

$$
-e^{2(v-a)}\left[\ddot{A}_\theta + (2\dot{a} + \dot{v} - \dot{b} - \dot{\varphi})\dot{A}_\theta\right] + A_\theta'' + (4a' - v' - b' - \varphi')A_\theta' = 0\,. \tag{2.5}
$$

The dilaton equation is:

$$
\begin{aligned}
-e^{2(v-a)}\left[\ddot{\varphi} + (2\dot{a} + \dot{v} + \dot{b})\dot{\varphi}\right] + \varphi'' + (4a' - v' + b')\varphi'& \\
-\frac{1}{2}\, e^{-2a+2v-2b-\varphi}(\dot{A}_\theta)^2 + \frac{1}{2}\, e^{-2b-\varphi}(A_\theta')^2 - \frac{2g^2}{\kappa^2}\, e^{2v+\varphi} &= 0\,,
\end{aligned}
\tag{2.6}
$$

The $(t\eta)$ Einstein equation is

$$3\,\dot{a}' + \dot{b}' - 3\,\dot{v}\,a' + \dot{b}\,b' - \dot{b}\,a' - \dot{v}\,b' + \dot{\varphi}\varphi' + e^{-2b-\varphi}\dot{A}_\theta A'_\theta = 0\,. \tag{2.7}$$

The (tt) Einstein equation is:

$$\begin{aligned}-e^{2(v-a)}\Big[3\,\ddot{a} + \ddot{b} + \ddot{v} + (\dot{v})^2 + (\dot{b})^2 + (\dot{\varphi})^2 - \dot{a}(\dot{v}+\dot{b})\Big] + a'' + a'(4\,a' - v' + b')\\ -\frac{3\kappa^2}{4}\,e^{-2a+2v-2b-\varphi}(\dot{A}_\theta)^2 - \frac{\kappa^2}{4}\,e^{-2b-\varphi}(A'_\theta)^2 + \frac{g^2}{\kappa^2}\,e^{2v+\varphi} = 0\,.\end{aligned} \tag{2.8}$$

The $(\eta\eta)$ Einstein equation is:

$$\begin{aligned}-e^{2(v-a)}\Big[\ddot{v} + \dot{v}(2\,\dot{a} + \dot{v} + \dot{b})\Big] + 4\,a'' + b'' + 4\,(a')^2 + (b')^2 - v'(4\,a' + b') + (\varphi')^2\\ +\frac{\kappa^2}{4}\,e^{-2a+2v-2b-\varphi}(\dot{A}_\theta)^2 + \frac{3\kappa^2}{4}\,e^{-2b-\varphi}(A'_\theta)^2 + \frac{g^2}{\kappa^2}e^{2v+\varphi} = 0\,.\end{aligned} \tag{2.9}$$

The $(\theta\theta)$ Einstein equation is:

$$\begin{aligned}-e^{2(v-a)}\Big[\ddot{b} + \dot{b}(2\,\dot{a} + \dot{v} + \dot{b})\Big] + b'' + b'(4\,a' - v' + b')\\ -\frac{3\kappa^2}{4}\,e^{-2a+2v-2b-\varphi}(\dot{A}_\theta)^2 + \frac{3\kappa^2}{4}\,e^{-2b-\varphi}(A'_\theta)^2 + \frac{g^2}{\kappa^2}\,e^{2v+\varphi} = 0\,.\end{aligned} \tag{2.10}$$

The (ij) Einstein equation is:

$$\begin{aligned}-e^{2(v-a)}\Big[\ddot{a} + \dot{a}(2\,\dot{a} + \dot{v} + \dot{b})\Big] + a'' + a'(4\,a' - v' + b')\\ +\frac{\kappa^2}{4}\,e^{-2a+2v-2b-\varphi}(\dot{A}_\theta)^2 - \frac{\kappa^2}{4}\,e^{-2b-\varphi}(A'_\theta)^2 + \frac{g^2}{\kappa^2}\,e^{2v+\varphi} = 0\,.\end{aligned} \tag{2.11}$$

2.3 Maximally symmetric 4D compactifications

Many explicit compactifications of the above field equations to four dimensions have been constructed over the years, starting 20 years ago with the Salam-Sezgin spherical solution [15]. These now include compactifications to flat 4D space on unwarped, rugby-ball solutions [1], as well as warped axially-symmetric internal dimensions having conical [2], and more general [19, 20, 21] singularities at the positions of two source branes. More recent generalizations have found warped de Sitter 4D geometries [22], as well as configurations for which the hyperscalars and 3-form fluxes are nontrivial [23].

Since we wish to perturb about the 4D flat solutions, we briefly summarize their properties here.[2] These solutions may be written

$$e^a = \mathcal{W}\,, \qquad e^v = \mathcal{A}\mathcal{W}^4 \qquad \text{and} \qquad e^b = \mathcal{A}\,, \tag{2.12}$$

[2]The conventions of ref. [19] may be obtained from ours by taking $R_{MN} \to -R_{MN}$, $\varphi \to -\varphi/2$ and $\kappa^2 = 1/2$, while those of [2] differ from those here only by the choice $\kappa^2 = 1$.

and so the bulk fields become

$$\begin{aligned}
\mathrm{d}s^2 &= \mathcal{W}^2(\eta)\,\eta_{\mu\nu}\,\mathrm{d}x^\mu \mathrm{d}x^\nu + \mathcal{A}^2(\eta)\Big[\mathcal{W}^8(\eta)\,\mathrm{d}\eta^2 + \mathrm{d}\theta^2\Big] \\
F_{\eta\theta} &= \left(\frac{q\mathcal{A}^2}{\mathcal{W}^2}\right) e^{-\lambda_3\eta} \qquad \text{and} \qquad e^{-\varphi} = \mathcal{W}^2\, e^{\lambda_3\eta}\,,
\end{aligned} \tag{2.13}$$

where

$$\begin{aligned}
\mathcal{W}^4 &= \left(\frac{\kappa^2 q\lambda_2}{2g\lambda_1}\right)\frac{\cosh[\lambda_1(\eta-\eta_1)]}{\cosh[\lambda_2(\eta-\eta_2)]} \\
\mathcal{A}^{-4} &= \left(\frac{2\kappa^2 q^3 g}{\lambda_1^3\lambda_2}\right) e^{-2\lambda_3\eta}\cosh^3[\lambda_1(\eta-\eta_1)]\cosh[\lambda_2(\eta-\eta_2)]\,,
\end{aligned} \tag{2.14}$$

where the field equations imply the constraint $\lambda_2^2 = \lambda_1^2 + \lambda_3^2$.

There are two particularly interesting special cases of these solutions. The first is obtained by taking $\lambda_3 = 0$, in which case the resulting geometries have purely conical singularities [20]. The second comes by taking both $\lambda_3 = 0$ and $\eta_1 = \eta_2$, in which case the conical geometry is also unwarped, since $\varphi = \varphi_0$ and $\mathcal{W} = \mathcal{W}_0$ are constants. In this latter case changing variables to proper distance, $\mathrm{d}\rho = \mathcal{A}\mathcal{W}^4\mathrm{d}\eta$, the Maxwell field becomes

$$F_{\rho\theta} = \partial_\rho A_\theta = \pm\frac{2g}{\kappa^2}\, e^{\varphi_0}\, B(\rho)\,, \tag{2.15}$$

where $B(\rho) \equiv e^{b(\rho)}$ satisfies

$$\frac{B''}{B} = b'' + (b')^2 = -\frac{4g^2}{\kappa^2}\, e^{\varphi_0}\,, \tag{2.16}$$

with solution $B(\rho) = B_0\sin(\rho/\rho_0)$ with $\rho_0 = \frac{1}{2}\kappa e^{-\varphi_0/2}/g$. These are the unwarped, rugby-ball generalizations [1] of the original spherical Salam-Sezgin solution [15] to include the back-reaction of two branes, located at the sphere's north and south poles.

In what follows we also use the extremely useful change of variables used in ref. [22],

$$\varphi = \frac{1}{2}\left(\mathcal{X} - \mathcal{Y} - 2\mathcal{Z}\right), \quad \log\mathcal{W} = \frac{1}{4}\left(\mathcal{Y} - \mathcal{X}\right) \;\text{ and }\; \log\mathcal{A} = \frac{1}{4}\left(3\mathcal{X} + \mathcal{Y} + 2\mathcal{Z}\right), \tag{2.17}$$

and so

$$e^{-\mathcal{X}} = e^{-\varphi/2}\mathcal{A}^{-1} = \frac{\kappa q}{\lambda_1}\cosh\left[\lambda_1(\eta-\eta_1)\right] \tag{2.18}$$

$$e^{-\mathcal{Y}} = e^{-\varphi/2}\mathcal{A}^{-1}\mathcal{W}^{-4} = \frac{2g}{\kappa\lambda_2}\cosh\left[\lambda_2(\eta-\eta_2)\right] \tag{2.19}$$

$$e^{-\mathcal{Z}} = e^{\varphi}\mathcal{W}^2 = e^{-\lambda_3\eta}\,. \tag{2.20}$$

It is straightforward to check that these are related by the Hamiltonian constraint (for evolution in the η direction):

$$-\frac{4g^2}{\kappa^2}\, e^{2\mathcal{Y}} + \kappa^2 q^2 e^{2\mathcal{X}} + \mathcal{X}'^2 - \mathcal{Y}'^2 + \mathcal{Z}'^2 = 0\,. \tag{2.21}$$

2.4 Asymptotic forms

The solutions to these equations describe geometries which typically become singular within the bulk. Because of the assumed axial symmetry this can occur at two (or fewer) points, which we can choose to place at $\eta = \pm\infty$. These are interpreted as being the positions of the 3-branes which source the corresponding field configuration. We require an explicit form for the asymptotic behaviour near these singularities in order to determine the boundary conditions in our later linearized perturbation analysis. We therefore pause here to outline what this asymptotic behaviour is.

If we use proper distance as the radial coordinate near the branes we have $\mathrm{d}s^2 = \hat{g}_{ab}\,\mathrm{d}x^a\mathrm{d}x^b + \mathrm{d}\rho^2$, and we imagine the brane position to be given by $\rho = 0$. For maximally-symmetric 4D geometries the asymptotic form of the bulk fields in the $\rho \to 0$ limit is then generically given by a power law form [22]

$$\begin{aligned} \mathrm{d}s^2 &\sim [c_a(H\rho)^\alpha]^2\, q_{\mu\nu}\,\mathrm{d}x^\mu\mathrm{d}x^\nu + \mathrm{d}\rho^2 + [c_\theta(H\rho)^\beta]^2\mathrm{d}\theta^2 \\ e^\varphi &\sim c_\phi(H\rho)^p \quad \text{and} \quad F^{\rho\theta} \sim c_f\,(H\rho)^\gamma\,, \end{aligned} \tag{2.22}$$

where α, β, p, γ, c_a, c_θ, c_ϕ and c_f are constants, H is an arbitrary dimensionful parameter, and $q_{\mu\nu}$ denotes the metric of 4D flat, de Sitter or anti-de Sitter space. For time-independent fields the field equations impose the following two Kasner-like constraints amongst the powers α, β, γ and p:

$$4\,\alpha^2 + \beta^2 + p^2 = 4\,\alpha + \beta = 1 \quad \text{and} \quad \gamma = p - 1\,. \tag{2.23}$$

Imposing these 3 conditions on the 4 powers leaves 1 undetermined, which we can choose to be p. Together with the prefactor, c_θ, this power can be related to the two physical choices which are available for the source branes: their tension and dilaton coupling [12, 20, 24, 25]. For example, explicit calculation using the solutions given above gives

$$\alpha_\pm = \frac{\lambda_2 - \lambda_1}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}\,, \quad \beta_\pm = \frac{\lambda_2 + 3\lambda_1 \mp 2\lambda_3}{5\lambda_2 - \lambda_1 \mp 2\lambda_3} \quad \text{and} \quad p_\pm = -\frac{2(\lambda_2 - \lambda_1 \mp 2\lambda_3)}{5\lambda_2 - \lambda_1 \mp 2\lambda_3}\,, \tag{2.24}$$

for the asymptotic form as $\eta \to \pm\infty$. As is easily verified, these satisfy eqs. (2.23) given above.

For time-dependent geometries it is useful to generalize the above to allow different asymptotics for the spatial and temporal components of the metric, replacing eq. (2.22) with:

$$\mathrm{d}s^2 \sim -[c_w(H\rho)^\omega]^2\,\mathrm{d}t^2 + [c_a(H\rho)^\alpha]^2\; g_{ij}\mathrm{d}x^i\mathrm{d}x^j + \mathrm{d}\rho^2 + [c_\theta(H\rho)^\beta]^2\mathrm{d}\theta^2\,, \tag{2.25}$$

with the powers now related by

$$\omega^2 + 3\,\alpha^2 + \beta^2 + p^2 = \omega + 3\,\alpha + \beta = 1 \quad \text{and} \quad \gamma = p - 1\,. \tag{2.26}$$

The additional parameter appearing in this asymptotic form is related to the freedom to separately specify the energy density and pressure on the source branes.

3. Linearization

In this section we set up the equations which govern axially-symmetric perturbations about the solutions described above which transform as scalars in 4D. The restriction to axially symmetric perturbations does not limit the ensuing stability analysis because the most unstable modes are generally the most symmetric, since any angular dependence contributes positively to the corresponding Kaluza-Klein mass.

3.1 Symmetries and gauge choices

The most general 4D scalar perturbations have the form

$$\begin{aligned} \mathrm{d}s^2 &= e^{2a}\Big[\eta_{\mu\nu} + M_{,\,\mu\nu}\Big]\mathrm{d}x^\mu \mathrm{d}x^\nu + N_{m,\mu}\,\mathrm{d}x^\mu \mathrm{d}x^m + g_{mn}\,\mathrm{d}x^m \mathrm{d}x^n \\ B &= B_{\mu\nu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^\nu + B_{m,\mu}\,\mathrm{d}x^\mu \wedge \mathrm{d}x^m + B_{mn}\,\mathrm{d}x^m \wedge \mathrm{d}x^n \\ \text{and} \qquad A &= \Omega_{,\mu}\,\mathrm{d}x^\mu + A_m\,\mathrm{d}x^m\,, \end{aligned} \tag{3.1}$$

where in four dimensions $B_{\mu\nu}$ dualizes to a 4D scalar ζ through a relation of the form $H_{\mu\nu\lambda} \propto \epsilon_{\mu\nu\lambda\rho}\partial^\rho\zeta$. We are free to use gauge symmetries to locally set $N_m = B_m = \Omega = 0$.

Discrete symmetry and mode mixing

As has been pointed out by earlier analyses [14, 26], for the purposes of a stability analysis it is not necessary to keep all of the remaining perturbations. To see why, notice that since our interest is in fluctuations depending only on t, $x^i$and η, the absence of a dependence on the coordinate θ implies also a symmetry under the reflection, $\theta \to -\theta$. If A_θ is nonzero in the background configuration (as is the case when a flux $F_{\eta\theta}$ is turned on in the extra dimensions), this is only a symmetry if we also independently require $A_M \to -A_M$.

This symmetry ensures that the linearized fluctuations can be divided into two classes, according to whether or not they are even or odd under these reflections. In particular we have

$$\begin{aligned} \{\delta a, \delta g_{\theta\theta}, \delta g_{\eta\eta}, \delta\varphi, \delta A_\theta\} &\qquad \text{(even)} \\ \{\delta g_{\eta\theta}, \delta\zeta, \delta B_{\eta\theta}, \delta A_\eta\} &\qquad \text{(odd)}\,. \end{aligned} \tag{3.2}$$

Since the symmetry guarantees that it is consistent with all of the equations of motion to set the odd fluctuations to vanish, these two categories of fluctuations cannot mix with one another at the linearized level. We take advantage of this fact to choose $\delta g_{\eta\theta} = \delta\zeta = \delta B_{\eta\theta} = \delta A_\eta = 0$. Again, the omission of these modes does not restrict the stability analysis which follows.

We are led in this way to the following ansatz for the metric and other bulk fields

$$\begin{aligned} \mathrm{d}s^2 &= e^{2a}\Big(\eta_{\mu\nu} + M_{,\,\mu\nu}\Big)\mathrm{d}x^\mu \mathrm{d}x^\nu + e^{2v}\,\mathrm{d}\eta^2 + e^{2b}\,\mathrm{d}\theta^2 \\ A &= A_\theta\,\mathrm{d}\theta \qquad \text{and} \qquad B = 0\,. \end{aligned} \tag{3.3}$$

To linearize the field equations about a static background we write

$$a(\eta,x) = a_0(\eta) + A(\eta,x)\,, \quad v(\eta,x) = v_0(\eta) + V(\eta,x)\,, \quad b(\eta,x) = b_0(\eta) + B(\eta,x)\,,$$
$$\varphi(\eta,x) = \varphi_0(\eta) + \Phi(\eta,x) \quad \text{and} \quad A_\theta(\eta,x) = a_\theta(\eta) + \mathcal{A}_\theta(\eta,x)\,, \tag{3.4}$$

where we allow the fluctuations to depend on all 4 coordinates, x^μ, and the background field configurations are given as above:

$$e^{a_0} = \mathcal{W}(\eta)\,, \quad e^{v_0} = \mathcal{A}(\eta)\mathcal{W}^4(\eta)\,, \quad e^{b_0} = \mathcal{A}(\eta)\,,$$
$$e^{-\varphi_0} = \mathcal{W}^2(\eta)\, e^{\lambda_3\eta} \quad \text{and} \quad a'_\theta = \frac{q\mathcal{A}^2(\eta)}{\mathcal{W}^2(\eta)}\, e^{-\lambda_3\eta}\,. \tag{3.5}$$

For some of the discussion to follow it is useful to choose proper distance computed using the background metric as a coordinate within the extra dimensions, as we did for the rugby ball solution above. We reserve the variable ρ for proper distance, and it is related explicitly to η by $\mathrm{d}\rho = e^{v_0(\eta)}\mathrm{d}\eta = \mathcal{A}\mathcal{W}^4\,\mathrm{d}\eta$. Notice that this choice of coordinates can be made independently of the gauge choice we make on the linearized fluctuations, which we now describe.

Three useful gauge choices

Finally, it is also not necessary to keep all of A, M, V and B independent, because one combination of these can be set to zero using an appropriate coordinate choice. There are three gauge choices which we use in what follows.

- *Comoving* (c) gauge: defined by the condition $\mathcal{A}_\theta^{(c)} = 0$, leading to

$$\mathrm{d}s^2 = e^{2a_0}e^{2A^{(c)}}\,\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{2v_0}e^{2V^{(c)}}\,\mathrm{d}\eta^2 + e^{2b_0}e^{2B^{(c)}}\,\mathrm{d}\theta^2 + 2N_{,\mu}\,\mathrm{d}\eta\mathrm{d}x^\mu\,, \tag{3.6}$$

  together with $A = a_\theta\mathrm{d}\theta$ and $\varphi = \varphi_0 + \Phi^{(c)}$. Notice that this gauge does not fall completely into the ansatz of eq. (2.4), because the coordinate transformation required to get into this form does not preserve the condition $\delta A_\theta = 0$. This is the origin of the new metric variable, $\delta g_{\eta\mu}$, in the above expression.

- *Gaussian Normal* (GN) gauge, defined by $V^{(GN)} = 0$ and so:

$$\mathrm{d}s^2 = e^{2a_0}\left(e^{2A^{(GN)}}\eta_{\mu\nu} + M^{(GN)}_{,\mu\nu}\right)\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{2b_0}e^{2B^{(GN)}}\mathrm{d}\theta^2 + \mathrm{d}\rho^2\,, \tag{3.7}$$

  together with $\varphi = \varphi_0 + \Phi^{(GN)}$ and $A_\theta = a_\theta + \mathcal{A}_\theta^{(GN)}$. It is obviously convenient also to use background-metric proper distance, ρ defined via $d\rho = e^{v_0}d\eta$.

- *Longitudinal* (l) gauge: defined by $M = 0$ and so

$$\mathrm{d}s^2 = e^{2a_0}e^{2A^{(l)}}\eta_{\mu\nu}\mathrm{d}x^\mu\mathrm{d}x^\nu + e^{2b_0}e^{2B^{(l)}}\mathrm{d}\theta^2 + e^{2V^{(l)}}\mathrm{d}\rho^2, \tag{3.8}$$

  together with $\varphi = \varphi_0 + \Phi^{(l)}$ and $A_\theta = a_\theta + \mathcal{A}_\theta^{(l)}$. Notice that we again choose to use background-metric proper distance, ρ, as the radial coordinate.

Which of these gauges is most convenient depends on the goal of the calculation. Comoving gauge is physically intuitive (since one follows perturbations along hypersurfaces of constant A_θ), and has the enormous benefit that the field equations completely decouple in this gauge, making the linearization analysis much easier. Longitudinal gauge is convenient for making contact with earlier calculations in the literature, since this is the gauge which has often been used. Finally, GN gauge is useful for analyzing the boundary conditions which fluctuations must satisfy near the source branes.

It is clearly useful to be able to transform between expressions obtained in these three gauges, so we next briefly summarize the main formulae which are required. To change from longitudinal to comoving gauge requires the transformation $x^M_{(l)} = x^M_{(c)} + \xi^M$, with $\xi^M = \varepsilon\,\delta^M_\rho$ so that $\rho_{(l)} = \rho_{(c)} + \varepsilon$, with $\varepsilon = \mathcal{A}^{(l)}_\theta / \partial_\rho a_\theta$ required to ensure $\mathcal{A}^{(c)}_\theta = 0$. This implies that the various fields in longitudinal gauge are related to the ones in comoving gauge by:

$$
\begin{aligned}
A^{(l)} = A^{(c)} + \varepsilon\,\partial_\rho a_0\,, \quad B^{(l)} = B^{(c)} + \varepsilon\,\partial_\rho b_0\,, \quad V^{(l)} = V^{(c)} + \partial_\rho \varepsilon\,, \\
N^{(c)} = -e^{4a_0+b_0}\varepsilon\,, \quad \Phi^{(l)} = \Phi^{(c)} + \varepsilon \partial_\rho \varphi_0\,.
\end{aligned}
\tag{3.9}
$$

Similarly, to go from comoving to GN gauge, we perform the transformations $\rho_{(c)} = \rho_{(GN)} + \varepsilon$ with $\partial_\rho \varepsilon = V^{(c)}$ chosen to ensure $V^{(GN)} = 0$. The further transformation, $x^\mu_{(c)} = x^\mu_{(GN)} + \partial^\mu \epsilon$, with $\partial_\rho \epsilon = e^{-(6a_0+b_0)} N^{(c)} - e^{-2a_0}\varepsilon$ is also required in order to ensure the vanishing of $N^{(GN)}$. Quantities in comoving gauge are then related to those in GN gauge by

$$
\begin{aligned}
A^{(c)} = A^{(GN)} + \varepsilon \partial_\rho a_0\,, \quad M^{(GN)} = -2\epsilon, \quad B^{(c)} = B^{(GN)} + \varepsilon \partial_\rho b_0 \\
\Phi^{(c)} = \Phi^{(GN)} + \varepsilon \partial_\rho \varphi_0, \quad \mathcal{A}^{(GN)}_\theta = -\varepsilon \partial_\rho a_\theta.
\end{aligned}
\tag{3.10}
$$

3.2 Linearized equations in comoving gauge

We now work out the perturbed equations in comoving gauge, since in this gauge we are able to decouple the fluctuations even when perturbing about non-conical background solutions. We use as the coordinate η in this section, with the derivative $\mathrm{d}/\mathrm{d}\eta$ denoted by primes.

We adopt the following notation for the perturbations $A^{(c)} = -\Psi/2$ and $V^{(c)} = \xi/2$, and so

$$
\mathrm{d}s^2 = e^{(\mathcal{Y}-\mathcal{X})/2} e^{-\Psi}\, \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{(3\mathcal{X}+\mathcal{Y}+2\mathcal{Z})/2}\left[e^{2(\mathcal{Y}-\mathcal{X})} e^{\xi}\, \mathrm{d}\eta^2 + e^{2B}\, \mathrm{d}\theta^2\right] + 2N_{,\mu} \mathrm{d}\eta\, \mathrm{d}x^\mu\,, \tag{3.11}
$$

while $e^\varphi = e^{(\mathcal{X}-\mathcal{Y}-2\mathcal{Z})/2} e^\Phi$ and $F_{\eta\theta} = a'_\theta = q e^{-\lambda_3 \eta} e^{2\mathcal{X}+\mathcal{Z}}$. In terms of these variables, and Fourier transforming with respect to time and space[3]so $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu = \omega^2$, the Maxwell equation is

$$
2\,\omega^2 e^{(\mathcal{X}-\mathcal{Y})/2} N + 2B' + 2\Phi' + \xi' + 4\Psi' = 0\,; \tag{3.12}
$$

[3]We emphasize that ω is the mass of the corresponding Kaluza-Klein mode and should be understood as the eigenvalue of the operator $\Box$. In this analysis, we do not make the assumption that the modes have a specific time-oscillatory behavior of the form $e^{ik_0 t}$. Thus this analysis includes both time-oscillating modes, and non-oscillating ones which correspond to zero-mass Kaluza Klein modes.

the dilaton equation is

$$\Phi'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\Phi - \frac{1}{2}\left(\kappa^2 q^2 e^{2\mathcal{X}} + \frac{4g^2}{\kappa^2}e^{2\mathcal{Y}}\right)\Phi = \varphi_0'\left(\frac{1}{2}\xi' + 2\Psi' - B' + \omega^2 e^{(\mathcal{X}-\mathcal{Y})/2}N\right) + \frac{2g^2}{\kappa^2}\, e^{2\mathcal{Y}}\xi + \kappa^2 q^2 e^{2\mathcal{X}} B\,; \quad (3.13)$$

the $(\mu\eta)$ Einstein equation is

$$2(2\mathcal{X}' + \mathcal{Z}')B - (2\mathcal{Y}' + \mathcal{Z}')\xi + 4\varphi_0'\Phi + 4B' - 6\Psi' = 0\,, \quad (3.14)$$

while the other off-diagonal term in the Einstein equation (the combination $(ii) - (jj)$, with $i \neq j$) is

$$e^{\frac{1}{2}(\mathcal{X}-5\mathcal{Y}-2\mathcal{Z})}\left[2N' + (\mathcal{X}' - \mathcal{Y}')N\right] - 2B - \xi + 2\Psi = 0. \quad (3.15)$$

A combination of the $(\eta\eta)$ Einstein equation, the dilaton equation of motion and the trace of the Einstein equation which involves only first derivatives with respect to η is

$$\frac{\omega^2}{2}\, e^{(\mathcal{X}-\mathcal{Y})/2}(2\mathcal{Y}' + \mathcal{Z}')N - \frac{2g^2}{\kappa^2}\, e^{2\mathcal{Y}}\xi + \frac{3\omega^2}{2}\, e^{2\mathcal{Y}+\mathcal{Z}}\Psi + (2\mathcal{Y}' + \mathcal{Z}')\Psi' - \frac{1}{2}\left(\kappa^2 q^2 e^{2\mathcal{X}} + \frac{4g^2}{\kappa^2}\, e^{2\mathcal{Y}}\right)\Phi + \varphi_0'\Phi' - \kappa^2 q^2 e^{2\mathcal{X}} B - \omega^2 e^{2\mathcal{Y}+\mathcal{Z}} B + (\mathcal{X}' - \mathcal{Y}')B' = 0\,. \quad (3.16)$$

The $(\theta\theta)$ Einstein equation is similarly

$$(2B'' + \Phi'') + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}(2B + \Phi) - 2\kappa^2 q^2 e^{2\mathcal{X}}(2B + \Phi) + 2\mathcal{X}'B' = \mathcal{X}'\left(4\Psi' + \xi' + 2\,\omega^2 e^{(\mathcal{X}-\mathcal{Y})/2}N\right), \quad (3.17)$$

and finally, the $(\mu\mu)$ Einstein equation is

$$\left(\Phi'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\Phi\right) - \left(\Psi'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\Psi - 2\mathcal{Z}'\Psi'\right) = \mathcal{Z}'\left(B' - \frac{1}{2}\,\xi' - \omega^2 e^{(\mathcal{X}-\mathcal{Y})/2}N\right). \quad (3.18)$$

We have here a set of seven equations for five variables (ξ, N, Ψ, B and Φ). Two of these equations appear as constraints and can be used to fix the lapse and shift functions: using the $(\mu\eta)$ equation (3.14) and the $(\eta\eta)$ equation (3.16), we can express both ξ and $\omega^2 N$ in terms of the three remaining variables Ψ, B and Φ, leading to five remaining equations for three variables. However these remaining equations are not independent since the Bianchi identities ensure that two of them are redundant. One can be expressed as a linear combination of the others, whilst the derivative of the Maxwell equation (3.12) is simply the derivative of (3.14) and a combination of the other equations.

We arrive in this way with three equations for three unknowns, which can be written as

$$B'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}B + \frac{1}{4}\left(6\mathcal{X}' + 2\mathcal{Y}' + 4\mathcal{Z}' + \frac{16\,g^2}{\kappa^2}\,\frac{e^{2\mathcal{Y}}}{2\mathcal{Y}' + \mathcal{Z}'}\right)B' + \frac{1}{8}\left(-12\kappa^2 q^2 e^{2\mathcal{X}} + \frac{16\,g^2}{\kappa^2}\, e^{2\mathcal{Y}}\frac{2\mathcal{X}' + \mathcal{Z}'}{2\mathcal{Y}' + \mathcal{Z}'}\right)B + \frac{1}{4}\left(3\mathcal{X}' + \mathcal{Y}' + 2\mathcal{Z}'\right)\Phi' + \frac{1}{16}\left(-12\kappa^2 q^2 e^{2\mathcal{X}} + \frac{16\,g^2}{\kappa^2}\, e^{2\mathcal{Y}}\frac{2\mathcal{X}' - 3\mathcal{Z}'}{2\mathcal{Y}' + \mathcal{Z}'}\right)\Phi - \frac{6\,g^2}{\kappa^2}\,\frac{e^{2\mathcal{Y}}}{2\mathcal{Y}' + \mathcal{Z}'}\Psi' = 0\,, \quad (3.19)$$

$$\Psi'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\Psi + \frac{12\, g^2}{\kappa^2}\,\frac{e^{2\mathcal{Y}}}{(2\mathcal{Y}'+\mathcal{Z}')}\Psi' + \frac{1}{2}(\mathcal{X}'-\mathcal{Y}')\left(\Phi'+2B'\right) - \frac{8\, g^2}{\kappa^2}\frac{e^{2\mathcal{Y}}}{2\mathcal{Y}'+\mathcal{Z}'}B' \tag{3.20}$$
$$-\frac{\kappa^2 q^2}{2}\, e^{2\mathcal{X}}\,(\Phi+2B) - \frac{2\, g^2}{\kappa^2}\,\frac{e^{2\mathcal{Y}}}{2\mathcal{Y}'+\mathcal{Z}'}\left[(2\mathcal{X}'-3\mathcal{Z}')\Phi + 2(2\mathcal{X}'+\mathcal{Z}')B\right] = 0\,,$$

and

$$\Phi'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\Phi + \varphi_0'(\Phi'+2B') + \frac{4\, g^2}{\kappa^2}\,\frac{e^{2\mathcal{Y}}}{2\mathcal{Y}'+\mathcal{Z}'}\left(3\Psi'-2B'\right) \tag{3.21}$$
$$-\frac{\kappa^2 q^2}{2}\, e^{2\mathcal{X}}\,(\Phi+2B) - \frac{2\, g^2}{\kappa^2}\,\frac{e^{2\mathcal{Y}}}{2\mathcal{Y}'+\mathcal{Z}'}\left[(2\mathcal{X}'-3\mathcal{Z}')\Phi + 2(2\mathcal{X}'+\mathcal{Z}')B\right] = 0\,.$$

These equations dramatically simplify once the change of variables $(\Phi, B, \Psi) \to \left(\tilde{f}, \tilde{\chi}, \tilde{\psi}\right)$ is performed, with

$$\begin{aligned}
\Phi &= \frac{2\mathcal{U}}{\kappa q}\, e^{-\mathcal{X}}\, \tilde{f} - 2B \\
B &= \frac{\mathcal{V}}{\sqrt{3}\mathcal{U}}\,\tilde{\chi} - \frac{1}{2}\,\Psi + \frac{2\mathcal{U}^2+\mathcal{X}'\mathcal{Z}'}{2\,\kappa q\,\mathcal{U}}\, e^{-\mathcal{X}}\,\tilde{f} \\
\Psi &= \frac{2\mathcal{Y}'+\mathcal{Z}'}{4\mathcal{V}}\tilde{\psi} + \frac{2\mathcal{U}^2-3\mathcal{Y}'\mathcal{Z}'}{4\sqrt{3}\mathcal{U}\mathcal{V}}\tilde{\chi} + \frac{\mathcal{U}^2-\mathcal{X}'\mathcal{Y}'}{2\,\kappa q\,\mathcal{U}}\, e^{-\mathcal{X}}\,\tilde{f}\,,
\end{aligned} \tag{3.22}$$

where use of the background field equations shows that the nominally field-dependent quantities $\mathcal{U}$ and $\mathcal{V}$ are really both constants:

$$\begin{aligned}
\mathcal{U}^2 &\equiv \mathcal{X}'^2 + \kappa^2 q^2 e^{2\mathcal{X}} = \lambda_1^2 \\
\mathcal{V}^2 &\equiv \mathcal{X}'^2 + \kappa^2 q^2 e^{2\mathcal{X}} + \frac{3}{4}\mathcal{Z}'^2 = \lambda_1^2 + \frac{3}{4}\lambda_3^2\,.
\end{aligned} \tag{3.23}$$

Notice that for conical backgrounds $\mathcal{U} = \mathcal{V} = \lambda_1$. In terms of these new variables the three field equations — eqs. (3.19), (3.20) and (3.21) — take the remarkably simple, decoupled form

$$\tilde{\chi}'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\tilde{\chi} = 0 \tag{3.24}$$
$$\tilde{f}'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\tilde{f} - \mathcal{U}^2\tilde{f} = 0 \tag{3.25}$$
$$\tilde{\psi}'' + \omega^2 e^{2\mathcal{Y}+\mathcal{Z}}\tilde{\psi} - \frac{32\, g^2}{\kappa^2}\,\frac{\mathcal{V}^2 e^{2\mathcal{Y}}}{(2\mathcal{Y}'+\mathcal{Z}')^2}\tilde{\psi} = 0\,. \tag{3.26}$$

Because these equations decouple from one another each can be solved separately, and presents no problem for numerical analysis. Exact solutions are also available in the case of backgrounds having conical singularities (including but not restricted to the rugby ball geometries), as shall be described in detail in later sections. Given the existence of variables for which the equations decouple in comoving gauge, one might wonder what the corresponding variables are in other gauges. These variables can be obtained by explicitly performing the required change of gauge, but because the results are not very simple we relegate their description to Appendix C.

It is noteworthy for later purposes that the decoupled equations (3.24 – 3.26) are independent of the background variable $\mathcal{X}$ (and hence also of the parameter η_1). Consequently the linearized equations of motion for the new variables $\tilde{\chi}, \tilde{f}$ and $\tilde{\psi}$ are identical for a general conical background geometry (for which $\mathcal{Z}' = 0$) and a rugby ball geometry (for which $\mathcal{Z}' = 0$ and $\mathcal{X}' = \mathcal{Y}'$). However once the solutions are inserted into the expressions for the metric fluctuations using relations (3.22), the results do depend on $\mathcal{X}$ and so the full fluctuations 'know' about the complete background geometry.

4. Properties of Solutions

Up to this point there has been no restriction on the form of the background geometry. In this section we specialize to special conical backgrounds for which it is possible to solve the above linearized equations in closed form. We then return to the more general non-conical geometry in order to discuss the asymptotic behaviour of the solutions in the near-brane limit. The section closes with an outline of how this asymptotic form is related to the physical properties of the branes which source the bulk geometries.

4.1 Analytic Solutions for General conical backgrounds

We now concentrate on the behaviour of the perturbations to solutions having only conical singularities, which are defined by the condition $\lambda_3 = 0$ (and so $\mathcal{Z}' \equiv 0$). Performing the change of variable $z = \tanh\left[\lambda_2 \left(\eta - \eta_2\right)\right]$, the perturbation equations become

$$\frac{\mathrm{d}^2\tilde{\chi}}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\tilde{\chi}}{\mathrm{d}z} + \left(\frac{\mu^2}{1-z^2}\right)\tilde{\chi} = 0 \tag{4.1}$$

$$\frac{\mathrm{d}^2\tilde{\psi}}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}z} + \left(\frac{1}{1-z^2}\right)\left(\mu^2 - \frac{2}{z^2}\right)\tilde{\psi} = 0 \tag{4.2}$$

$$\frac{\mathrm{d}^2\tilde{f}}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\tilde{f}}{\mathrm{d}z} + \left(\frac{1}{1-z^2}\right)\left(\mu^2 - \frac{1}{1-z^2}\right)\tilde{f} = 0, \tag{4.3}$$

with $\mu^2 = \omega^2\kappa^2/4g^2$, and ω is the 4D mode mass. Notice that these equations are symmetric under $z \to -z$, which corresponds to reflections of η around $\eta = \eta_2$. The reason for this symmetry is clear in the special case of the rugby-ball solutions (for which $\eta_1 = \eta_2$), since it then corresponds to reflections of the spherical geometry about its equator. In the more general warped conical geometries this symmetry instead follows from the above-mentioned circumstance that the equations are completely independent of η_1, and so take the same form for general conical geometries as they do for the rugby ball.

Defining $\nu = \frac{1}{2}\left[-1 + \sqrt{1+4\mu^2}\right]$ — or equivalently writing $\mu^2 = \nu(\nu+1)$ — the solutions to these equations become

$$\begin{aligned} \tilde{\chi} &= C_1\, P_\nu(z) + C_2\, \mathrm{Re}\, Q_\nu(z) \\ &= C_1 F\left[-\nu, 1+\nu; 1; \frac{1}{2}(1-z)\right] + C_2\sqrt{\pi}\,\frac{\Gamma(\nu+1)}{\Gamma(\nu+\frac{3}{2})}\,\mathrm{Re}\,\frac{1}{(2z)^{\nu+1}}\, F\left[\frac{1}{2}(1+\nu), 1+\frac{\nu}{2}; \nu+\frac{3}{2}; z^{-2}\right] \end{aligned} \tag{4.4}$$

$$\tilde{\psi} = C_3\, z^{-1} F\left[-\frac{1}{2}(1+\nu), \frac{\nu}{2}; -\frac{1}{2}; z^2\right] + C_4\, z^2 F\left[1-\frac{\nu}{2}, \frac{1}{2}\,(3+\nu)\,; \frac{5}{2}; z^2\right] \tag{4.5}$$

$$\tilde{f} = \frac{1}{\sqrt{1-z^2}}\left(C_5\, F\left[\frac{\nu}{2}, -\frac{1}{2}(1+\nu); \frac{1}{2}; z^2\right] + C_6\ z\ F\left[\frac{1}{2}(1+\nu), -\frac{\nu}{2}; \frac{3}{2}; z^2\right]\right), \tag{4.6}$$

where the C_i, $i = 1,..,6$ are integration constants and $F(a,b;c;z)$ is the hypergeometric function. Neglecting the overall normalization of all three modes, we see that the above expressions provide a 9-parameter family of solutions. Notice also that these solutions can be decomposed into symmetric and anti-symmetric combinations under the symmetry $z \to -z$, and this is already explicit for the functions $\tilde{\psi}$ and $\tilde{f}$.

We see that the mode energies satisfy

$$\omega^2 = \frac{\nu(\nu+1)}{r_0^2}, \tag{4.7}$$

with $1/r_0 = 2g/\kappa$, and so the spectrum depends on the values which are allowed for the parameter ν. Notice that for real ν we have $\omega^2 \geq 0$ except for the interval $-1 < \nu < 0$. As usual, the values which are allowed for ν are related to the behaviour which is demanded of the solutions at the boundaries, which in the present instance corresponds to the near-brane limits $z \to \pm 1$. For this reason we next explore the asymptotic form taken by the solutions in the near-brane limit.

4.2 Asymptotic Forms

We next examine three aspects of the near-brane limit. In this section we first explore the asymptotic forms taken by the solutions just constructed for fluctuations about backgrounds having only conical singularities. We then see how these asymptotic forms generalize to the broader situation where the background has more generic singularities. A preliminary connection is drawn in the next section between this asymptotic behaviour and the properties of the source branes.

Conical Backgrounds

The asymptotic form of the solutions found above near the brane positions at $z = \pm 1$ may be found by using the properties of the Hypergeometric functions given in appendix D. This shows that for generic ν, $\tilde{\chi}$ and $\tilde{\psi}$ diverge logarithmically as one approaches the branes, whereas $\tilde{f}$ generically diverges as $(1-z^2)^{-1/2} \sim \cosh[\lambda_2(\eta - \eta_2)] \sim \rho^{-1}$.

As usual, less singular asymptotic behaviour can be possible for those ν for which the hypergeometric series terminates, which for $F(a,b;c;z)$ occurs when either a or b vanishes. Inspection of the explicit solutions then shows that termination happens when ν is quantized to be a non-negative integer: $\nu = \ell = 0, 1, 2, \ldots$. Notice that the resulting quantization of KK mass, from eq. (4.7), implies in this case $\omega^2 \geq 0$.

For $\nu = \ell$ one – but not both – of the two Hypergeometric functions appearing for each of $\tilde{\chi}$, $\tilde{\psi}$ and $\tilde{f}$ terminates, allowing a less singular near-brane limit to be obtained through an appropriate choice for the integration constants C_i. For instance, if we require — as we shall argue on the grounds of normalizability below — that all fluctuations must be less singular than $(1-z^2)^{-1}$, then this is possible for $\tilde{f}$ only if $\nu = \ell$ and one of C_5 or C_6 vanishes (which one depends on whether ℓ is even or odd).[4] Once this has been done the functions $\tilde{\chi}$ and $\tilde{\psi}$ can still diverge logarithmically as $z \to \pm 1$, leaving a 4-parameter family of potentially logarithmically singular solutions (up to overall normalization).

Completely regular solutions are also possible when $\nu = \ell$, provided that one of C_1 or C_2 and one of C_3 or C_4 also vanishes. For instance, $\tilde{\chi}$ is regular everywhere if $\nu = \ell$ and $C_2 = 0$ since in this case the solution P_ℓ degenerates to the Legendre polynomials. This leaves a 2-parameter family of nonsingular solutions (up to overall normalization).

Non-conical backgrounds

Although it is difficult to find explicit solutions in the generic case where the background has non-conical singularities. It is possible to use the linearized equations to estimate the asymptotic behaviour of the mode functions as $\eta \to \pm\infty$ (or $z \to \pm 1$). Repeating the steps which led to the equations of the previous section for this more general case, we instead find

$$\frac{\mathrm{d}^2\tilde{\chi}}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\tilde{\chi}}{\mathrm{d}z} + \frac{\tilde{\mu}^2}{(1-z^2)}\left(\frac{1+z}{1-z}\right)^{\delta\lambda}\tilde{\chi} = 0 \tag{4.8}$$

$$\frac{\mathrm{d}^2\tilde{\psi}}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\tilde{\psi}}{\mathrm{d}z} + \left(\frac{1}{1-z^2}\right)\left[\frac{\tilde{\mu}^2}{(1-z^2)}\left(\frac{1+z}{1-z}\right)^{\delta\lambda} - \frac{2\left(1-\delta\lambda^2\right)}{(z-\delta\lambda)^2}\right]\tilde{\psi} = 0 \tag{4.9}$$

$$\frac{\mathrm{d}^2\tilde{f}}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\tilde{f}}{\mathrm{d}z} + \left(\frac{1}{1-z^2}\right)\left[\frac{\tilde{\mu}^2}{(1-z^2)}\left(\frac{1+z}{1-z}\right)^{\delta\lambda} - \frac{1-4\delta\lambda^2}{1-z^2}\right]\tilde{f} = 0, \tag{4.10}$$

with $\tilde{\mu}^2 = \mu^2\, e^{\lambda_3 \eta_2}$ and $\delta\lambda = \lambda_3/2\lambda_2$.

Since $|\delta\lambda| < 1/2$ the term in eq. (4.8) which is proportional to $\tilde{\chi}$ remains subdominant compared to the terms in $\left(1-z^2\right)^{-1}\mathrm{d}\tilde{\chi}/\mathrm{d}z$ and $\mathrm{d}^2\tilde{\chi}/\mathrm{d}z^2$, such that the asymptotic behaviour of $\tilde{\chi}$ still remains logarithmic near the brane. Similar arguments show that $\tilde{\psi}$ and $\tilde{f}$ also keep the same asymptotic behaviour as in the conical case.

Normalizability

A restriction on the asymptotic forms follows from the condition that the mode functions be normalizable, as we now explore. As we show in detail in the next section, the equations of motion for each of the decoupled modes can be derived from a reduced action, eq. (5.11). In this action all modes, u, have the same kinetic term, $e^{2\mathcal{Y}+\mathcal{Z}} u \Box u$, where $e^{2\mathcal{Y}+\mathcal{Z}} = \mathcal{A}^2\mathcal{W}^6$, and so the definition of the inner product can be taken to be the same for all three of $\tilde{\chi}, \tilde{\psi}$ and $\tilde{f}$, having the form:

$$\langle u, v\rangle = \frac{i}{2}\int_\Sigma \mathrm{d}^3x\,\mathrm{d}\theta\,\mathrm{d}\eta\,\mathcal{A}^2\mathcal{W}^6\left(u\partial_t v^* - u^*\partial_t v\right),$$

[4] We give explicit expressions for some of the lowest of these nonsingular modes for $\tilde{\psi}$ and $\tilde{f}$ in Appendix A for longitudinal gauge.

for any two modes u and υ. Here Σ denotes a surface of constant t. The norm of a mode is therefore defined by

$$\langle u, u\rangle = \omega \int_\Sigma \mathrm{d}^3x\, \mathrm{d}\theta\, \mathrm{d}\eta\, \mathcal{A}^2\mathcal{W}^6 uu^* = \omega \int_\Sigma \mathrm{d}^3x\, \mathrm{d}\theta\, \mathrm{d}\rho\, \mathcal{A}\mathcal{W}^2 uu^*, \tag{4.11}$$

where we use $\mathrm{d}\rho/\mathrm{d}\eta = \mathcal{A}\mathcal{W}^4$ and $\partial_t u = -i\omega u$.

Our interest is in the conditions placed on the asymptotic behaviour of linearized modes by the convergence of these integrals. Given the asymptotic form, eq. (2.22), of the background metric, with powers α_0, β_0, p_0 *etc.*, we have $\mathcal{A}\mathcal{W}^2 \propto \rho^{\beta_0+2\alpha_0} \propto \rho^{1-2\alpha_0}$ in the near-brane limit ($\rho \to 0$). In order to be normalizable, modes should therefore vary as $u \propto \rho^\upsilon$ with power $\upsilon > -1+\alpha_0$ as $\rho \to 0$. In particular, for a conical background with $\alpha_0 = 0$ and $\beta_0 = 1$, modes must diverge less rapidly than $1/\rho$ near both branes.

4.3 Boundary conditions and brane properties

The above considerations show that the stability of the linearized fluctuations is related to the behaviour of the modes as they approach the branes. In this section we briefly explore to what extent the properties of the linearized solutions capture what we know about the asymptotic properties of the exact solutions. Exploring this connection also allows us to make some contact between the conditions for stability and the physical properties of the branes which source the bulk field configurations, through considerations along the lines of those given in refs. [12, 20, 24].

Asymptotics in GN gauge

Recall for these purposes that on general grounds the various bulk fields are known (in GN gauge) to approach the branes through a power-law form given by eqs. (2.22). In principle there are two cases to consider, depending on whether or not the background and perturbed fields share the same powers in these asymptotic expressions. That is, if we imagine that the background and perturbed metric components, e^{a_0} and $e^a = e^{a_0}e^A$, satisfy

$$e^{a_0} \to c_{a0}(H\rho)^{\alpha_0} \quad \text{and} \quad e^a \to c_a(H\rho)^\alpha\,, \tag{4.12}$$

near $\rho = 0$, then we see that the fluctuation field, A, must satisfy

$$A \to (\alpha - \alpha_0)\,\ln(H\rho) + \ln\left(\frac{c_a}{c_{a0}}\right) + \cdots\,, \tag{4.13}$$

in the same limit. Similar expressions also hold for the fields B, V, and Φ.

Since the Maxwell field strength varies as $F_{\rho\theta} \propto \rho^{\gamma+2\beta} \propto \rho^{p+2\beta-1}$ the same argument has slightly different implications for $\mathcal{A}_\theta$, since $a_\theta \propto \rho^{p_0+2\beta_0}$ and $A_\theta = a_\theta + \mathcal{A}_\theta \propto \rho^{p+2\beta}$. (Notice that this gives the expected results $F_{\rho\theta} \propto \rho$ and $A_\theta \propto \rho^2$ in the conical case, for which $\beta = 1$ and $p = 0$.) In this case we have $\mathcal{A}_\theta \propto \rho^{p+2\beta}$ if $p + 2\beta < p_0 + 2\beta_0$, because the small-$\rho$ behaviour of A_θ dominates that of a_θ. Things are different if $p + 2\beta > p_0 + 2\beta_0$, however, since in this case a_θ dominates A_θ, and in order to achieve this $\mathcal{A}_\theta$ must contain a piece which varies in the same way and cancels the contribution of a_θ as $\rho \to 0$. As a consequence in this case $\mathcal{A}_\theta \propto \rho^{p_0+2\beta_0}$ for small ρ.

How singular this is depends on the allowed range of $p+2\beta$. Notice that in general the constraints $4\alpha+\beta = 4\alpha^2+\beta^2+p^2 = 1$ require $\frac{5}{4}\beta^2 - \frac{1}{2}\beta + p^2 = \frac{3}{4}$. Solving this equation for p, the sum $p+2\beta$ is thus minimal ('$-$' sign) and maximal ('$+$' sign) for $\beta = \frac{1}{5} \mp \frac{16}{5\sqrt{21}}$, and $p = \mp\frac{2}{\sqrt{21}}$, so that the sum is in the range $-1.433 \simeq \frac{2}{5}(1-\sqrt{21}) \le p+2\beta \le \frac{2}{5}(1+\sqrt{21}) \simeq 2.233$. We see that perturbations to $\mathcal{A}_\theta$ can become quite singular. In the particular case of conical singularities $p=0$ and $\beta=1$ and so $p+2\beta=2$, leading to smooth behaviour near the branes.

We see from the above that the perturbations need not be smooth at the brane positions, but that in GN gauge the metric perturbations can at worst diverge logarithmically in the near-brane limit. (Perturbations to the Maxwell field can be more singular than logarithmic, but only if p and β are sufficiently negative.) Since the arguments of refs. [12, 20, 24] relate powers like α to physical properties on the branes, these logarithmically singular perturbations only arise if some of the brane properties are themselves perturbed. It is only in the limit that the asymptotic near-brane behaviour is the same before and after perturbation that the perturbed solutions are nonsingular at the brane positions. This is the case in particular if it is assumed that the geometry has purely conical singularities before and after perturbation, or if the sources are represented as delta functions with the implicit associated assumption that the bulk fields are well-defined when evaluated at the brane positions – as is common in higher-codimension stability analyses, such as that of ref. [14].

Asymptotic forms in comoving gauge

Although we have seen that within GN gauge fluctuations diverge at most logarithmically near the source branes, we next determine what this implies for the near-brane behaviour in the (comoving) gauge we use in our analysis. In this section we examine in particular the near-brane asymptotics in comoving gauge.

Choosing background proper distance, ρ, (rather than η) as the radial coordinate we have seen that the perturbations in comoving and GN gauge are related by eqs. (3.10), which we reproduce here for ease of reference:

$$\begin{aligned} A^{(c)} = A^{(GN)} + \varepsilon\partial_\rho a_0\,, \quad B^{(c)} = B^{(GN)} + \varepsilon\partial_\rho b_0 \\ V^{(c)} = \partial_\rho\varepsilon \quad \text{and} \quad \Phi^{(c)} = \Phi^{(GN)} + \varepsilon\partial_\rho\varphi_0\,, \end{aligned} \tag{4.14}$$

with $\varepsilon = -\mathcal{A}_\theta^{(GN)}/(\partial_\rho a_\theta)$. The condition $M^{(c)} = 0$ is ensured through a change of the coordinates x^μ with parameter $\epsilon = -M^{(GN)}/2$.

Since we know how the fluctuations vary with ρ as $\rho \to 0$ within GN gauge, the above equations allow this information to be carried over to comoving gauge. In particular, since $a_\theta + \mathcal{A}_\theta \propto \rho^{p+2\beta}$ and $a_\theta \propto \rho^{p_0+2\beta_0}$, we saw $\mathcal{A}_\theta \propto \rho^\zeta$, where $\zeta = \min(p+2\beta, p_0+2\beta_0)$. We see that $\varepsilon \propto \rho^{1+\Delta}$ and that there are two cases: (i) $\Delta = (p-p_0)+2(\beta-\beta_0)$ if $p+2\beta < p_0+2\beta_0$, or (ii) $\Delta = 0$ if $p+2\beta \ge p_0+2\beta_0$. Since a_0, b_0, φ_0, $A^{(GN)}$, $B^{(GN)}$ and $\Phi^{(GN)}$ all vary at most logarithmically for small ρ, we see from the above that all of the fluctuations in comoving gauge are also at worst logarithmic provided only that $\Delta = (p+2\beta) - (p_0 + 2\beta_0) \geq 0$. (In particular, for conical backgrounds the perturbations are at most logarithmic provided $\Delta = (p + 2\beta - 2) \geq 0$.)

These arguments show that the fluctuations Φ, B and Ψ diverge at most logarithmically within comoving gauge for a broad range of physical situations. Inspection of the definitions, eqs. (3.24)–(3.26), shows that this implies the same conclusion for $e^{-\mathcal{X}}\tilde{f}$, $\tilde{\chi}$ and $\tilde{\psi}$.

5. General Stability Analysis

We now give several general arguments in favour of stability for a broad class of boundary conditions. We provide two types of arguments, which complement one another. The first of these works directly with the linearized equations derived above, and can be used directly with either the longitudinal-gauge (provided in Appendix (A)) – or comoving-gauge formulations. The second argument is instead cast in terms of the action, and closes a loophole left by the previous equation-of-motion analysis. We make this second argument only for comoving gauge (and not for longitudinal gauge, say) due to complications which arise in constructing the relevant action in other gauges.

5.1 Equation of motion and tachyons

We start by arguing for stability directly with the equations of motion. The goal of this argument is to relate the sign of the energy eigenvalue, ω^2, to the boundary conditions which the fluctuations satisfy near the positions of the source branes.

The most direct way to do so is to multiply eq. (3.24) by χ^*; sum the result with its complex conjugate; and integrate the answer over the extra dimensions, to get:

$$\omega^2 \int_{-\infty}^{\infty} \mathrm{d}\eta \, e^{2\mathcal{Y}+\mathcal{Z}} \, |\tilde{\chi}|^2 = -\frac{1}{2}\Big[(|\tilde{\chi}|^2)'\Big]_{-\infty}^{\infty} + \int_{-\infty}^{+\infty} \mathrm{d}\eta \, |\tilde{\chi}'|^2 \,. \tag{5.1}$$

This shows how the sign of ω^2 is related to the behaviour of $\tilde{\chi}$ near the two brane positions. Notice that the combination $e^{2\mathcal{Y}+\mathcal{Z}}$ appearing on the left-hand side of (5.1) is related to $\mathcal{A}$ and $\mathcal{W}$ by $e^{2\mathcal{Y}+\mathcal{Z}} = \mathcal{A}^2\mathcal{W}^6$.

Although we do not have explicit solutions to the linearized equations for general non-conical backgrounds, the asymptotic behaviour of these solutions was argued above to be the same as in the conical case. In particular, both fluctuations $\tilde{\chi}$ and $\tilde{\psi}$ are typically proportional to $\log\rho \sim \eta$ as $\eta \to \pm\infty$, whereas $\tilde{f}$ usually diverges as $1/\rho \sim e^{|\eta|}$ in this limit. Thus the general asymptotic behaviour as $\eta \to \pm\infty$ is

$$\begin{aligned} &\tilde{\chi} \sim (C_\pm \eta + D_\pm) \,, \qquad \tilde{\psi} \sim (E_\pm \eta + F_\pm) \,, \\ &\qquad \text{and} \quad \tilde{f} \sim \Big(G_\pm \, e^{\lambda_2|\eta|} + H_\pm \, e^{-\lambda_2|\eta|} \Big) \,, \end{aligned} \tag{5.2}$$

for constants $C_\pm$, $D_\pm$, $E_\pm$, $G_\pm$ and $H_\pm$. Using this in eq. (5.1), and cutting off the integrations at $\eta = \pm\Lambda$, gives

$$\omega^2 \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, e^{2\mathcal{Y}+\mathcal{Z}} \, |\tilde{\chi}|^2 = -\left(|C_+|^2 + |C_-|^2\right)\Lambda - \mathrm{Re}\left(D_+^* C_+ - D_-^* C_-\right) + \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, |\tilde{\chi}'|^2 \,. \quad (5.3)$$

We see from this that if the fluctuation is required to remain finite on both branes — *i.e.* $C_+ = C_- = 0$ — the boundary term vanishes and the squared energy of any mode is necessarily positive: $\omega^2 \geq 0$. Furthermore, ω only vanishes if $\tilde{\chi}$ is a constant throughout the entire bulk.

Applying the same argument to the $\tilde{\psi}$ equation gives:

$$\begin{aligned}\omega^2 \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, e^{2\mathcal{Y}+\mathcal{Z}} \, |\tilde{\psi}|^2 &= -\frac{1}{2}\Big[(|\tilde{\psi}|^2)'\Big]_{-\Lambda}^{+\Lambda} + \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, \left(|\tilde{\psi}'|^2 + \frac{32\, g^2}{\kappa^2} \frac{\mathcal{V}^2 e^{2\mathcal{Y}}}{(2\mathcal{Y}' + \mathcal{Z}')^2} \, |\tilde{\psi}|^2 \right) \\ &= -\left(|E_+|^2 + |E_-|^2\right)\Lambda - \mathrm{Re}\left(F_+^* E_+ - F_-^* E_-\right) \\ &\quad + \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, \left(|\tilde{\psi}'|^2 + \frac{32\, g^2}{\kappa^2} \frac{\mathcal{V}^2 e^{2\mathcal{Y}}}{(2\mathcal{Y}' + \mathcal{Z}')^2} \, |\tilde{\psi}|^2 \right) .\end{aligned} \quad (5.4)$$

Here again, $\omega^2 \geq 0$ if $E_+ = E_- = 0$ is imposed so that the fluctuation remains finite on both branes, and can only vanish if $\tilde{\psi}$ vanishes identically.

Finally, the same argument applied to the mode $\tilde{f}$ gives

$$\begin{aligned}\omega^2 \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, e^{2\mathcal{Y}+\mathcal{Z}} \, |\tilde{f}|^2 &= -\frac{1}{2}\Big[(|\tilde{f}|^2)'\Big]_{-\Lambda}^{+\Lambda} + \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, \left(|\tilde{f}'|^2 + \mathcal{U}^2 \, |\tilde{f}|^2 \right) \\ &= -\lambda_2 \left(|G_+|^2 + |G_-|^2\right) e^{2\lambda_2 \Lambda} + \lambda_2 \left(|H_+|^2 + |H_-|^2\right) e^{-2\lambda_2 \Lambda} \\ &\quad + \int_{-\Lambda}^{+\Lambda} \mathrm{d}\eta \, \left(|\tilde{f}'|^2 + \mathcal{U}^2 \, |\tilde{f}|^2 \right) ,\end{aligned} \quad (5.5)$$

so that finiteness of the boundary terms requires $G_\pm = 0$ if $\lambda_2 > 0$ or $H_\pm = 0$ if $\lambda_2 < 0$. In either case $\omega^2 \geq 0$ and can only vanish if $\tilde{f} \equiv 0$ throughout the bulk.

In this way we see that perturbations about any background (let it be conical or not) are marginally stable, provided that the fluctuations are restricted to be nonsingular at the brane positions.

5.2 Action analysis and ghosts

The previous argument shows that the eigenmode frequency ω must be real for a broad choice of boundary conditions at the brane positions. However one might worry that the Lagrangian density for the linearized fluctuation might have the form $\mathcal{L} = \Phi^* f(\varphi)[\Box - g(\varphi)]\Phi$, where φ (resp. Φ) denotes a generic background (resp. fluctuation) field, and $f(\varphi)$ and $g(\varphi)$ are background-field dependent quantities. Notice that the equation of motion for Φ implies $[-\Box + g(\varphi)]\Phi = 0$, and so implies $\omega^2 \geq 0$ if $g(\varphi) \geq 0$. But this might still imply a negative contribution to the fluctuation energy if the prefactor $f(\varphi)$ should happen to be negative for some configurations φ. (Indeed, precisely this form of instability was argued in ref. [14] to occur in 6D chiral supergravity linearized about rugby-ball configurations due to kinetic-term mixing amongst various KK modes.) We next present a second argument for stability, which closes this particular loophole.

The starting point in this approach is to expand the action, eq. (2.1), to second order in the perturbations. Working in comoving gauge and using the coordinate η we get in this way the following quadratic action

$$S^{(2)} = \int \mathrm{d}^6 x \left(\mathcal{L}_{\rm kin} + \mathcal{L}_{\rm dyn}\right), \tag{5.6}$$

$$\mathcal{L}_{\rm kin} = \frac{1}{4\kappa^2} e^{\frac{1}{2}(\mathcal{X}-\mathcal{Y})} \Big[6\Psi' - 4B' - 4\varphi_0'\Phi - 2(2\mathcal{X}' + \mathcal{Z}')B + (2\mathcal{Y}' + \mathcal{Z}')\xi \Big] \Box N \tag{5.7}$$
$$+ \frac{3}{4\kappa^2} e^{2\mathcal{Y}+\mathcal{Z}} \left(2B - \Psi\right) \Box\Psi + \frac{1}{4\kappa^2} e^{2\mathcal{Y}+\mathcal{Z}} (3\Psi - 2B)\Box\xi + \frac{1}{2\kappa^2} e^{2\mathcal{Y}+\mathcal{Z}} \Phi\Box\Phi$$

$$\mathcal{L}_{\rm dyn} = \frac{1}{2\kappa^2}\left(3\Psi' - 4B'\right)\Psi' + \frac{1}{\kappa^2}(2\mathcal{X}' + \mathcal{Z}')\Psi B' - \frac{1}{2\kappa^2}\Phi'^2 - \frac{1}{2\kappa^2}(\mathcal{Y}' - \mathcal{X}')\xi B' \tag{5.8}$$
$$- \frac{1}{2\kappa^2}\left(4\Psi' - 2B' + \xi'\right)\varphi_0'\Phi + \frac{1}{2\kappa^2}(2\mathcal{Y}' + \mathcal{Z}')\xi\Psi' - \frac{q^2}{4} e^{2\mathcal{X}}\Phi^2$$
$$- \frac{q^2}{2} e^{2\mathcal{X}} B\left(B + \xi + 4\Psi + 2\Phi\right) - \frac{g^2}{\kappa^4} e^{2\mathcal{Y}}\left(\Phi + 2\xi\right)\Phi - \frac{g^2}{2\kappa^4} e^{2\mathcal{Y}}\xi^2 ,$$

where we allow the fluctuations to depend on all four noncompact coordinates, x^μ, $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$ denotes the flat d'Alembertian, and the split into 'dyn' and 'kin' distinguishes terms which involve $\Box$ from those which do not. We obtain the above expression by freely integrating by parts while ignoring the surface terms which this procedure introduces at the brane positions. This neglect of surface terms is justified inasmuch as our goal is to exclude the possibility of the negative (ghost-like) kinetic terms, as described above.

Differentiating this action with respect to the shift function N leads to the constraint

$$\xi = \frac{2(2\mathcal{X}' + \mathcal{Z}')B + 4\varphi_0'\Phi + 2\left(2B' - 3\Psi'\right)}{2\mathcal{Y}' + \mathcal{Z}'}, \tag{5.9}$$

while differentiating with respect to the lapse function ξ gives the constraint for N:

$$\Box N = \frac{e^{\frac{1}{2}(\mathcal{Y}-\mathcal{X})}}{2\mathcal{Y}' + \mathcal{Z}'} \Bigg[\kappa^2 q^2 e^{2\mathcal{X}}\left(2B + \Phi\right) + \frac{4\,g^2}{\kappa^2} e^{2\mathcal{Y}}\left(\xi + \Phi\right) + e^{2\mathcal{Y}+\mathcal{Z}}\Box\left(2B - 3\Psi\right) \tag{5.10}$$
$$+ 2(\mathcal{Y}' - \mathcal{X}')B' - 2(2\mathcal{Y}' + \mathcal{Z}')\Psi' - 2\varphi_0'\Phi' \Bigg].$$

These two constraints are consistent with what was obtained in eqs. (3.14) and (3.16). As before, we may use both these constraints to obtain the equations of motion for the three remaining degrees of freedom, Φ, B and Ψ. Substituting eqs. (5.9) and (5.10) into the action, together with the change of variable (3.22), the resulting action takes the remarkably simple form

$$S^{(2)} = \frac{1}{\kappa^2}\int \mathrm{d}^6 x \Bigg\{ \tilde{\chi}\Big[\partial_\eta^2 + e^{2\mathcal{Y}+\mathcal{Z}}\Box\Big]\tilde{\chi} + \tilde{\psi}\left[\partial_\eta^2 + e^{2\mathcal{Y}+\mathcal{Z}}\Box - \frac{32\,g^2}{\kappa^2}\,\frac{\mathcal{V}^2 e^{2\mathcal{Y}}}{(2\mathcal{Y}' + \mathcal{Z}')^2}\right]\tilde{\psi}$$
$$+ \tilde{f}\Big[\partial_\eta^2 + e^{2\mathcal{Y}+\mathcal{Z}}\Box - \mathcal{U}^2\Big]\tilde{f} \Bigg\}. \tag{5.11}$$

Varying this action gives the same equations as the ones obtained in (3.24, 3.25, 3.26), confirming that the substitution of ansatz (3.6) into the action consistently reproduces the equations of motion of the full theory.

Since all of the kinetic terms in the action are positive, it follows immediately that the theory has no ghost modes (for which $f(\varphi) < 0$). Because this disagrees with the conclusions drawn in [14], we also reproduce their longitudinal-gauge analysis in Appendix B and identify a sign error which we believe to be the source of the discrepancy.

6. Conclusions

In this paper we have studied the linearized evolution of perturbations to gauged chiral 6D supergravity compactified to 4D on a broad class of static and axially-symmetric vacuum solutions, sourced by two space-filling 3-branes. We follow previous authors in focussing on scalar modes which share the axial symmetry of the background, and which are even under a convenient parity transformation. These restrictions are consistent with the equations of motion, since they are enforced by symmetries. They do not compromise the stability conclusions because the excluded modes necessarily have higher squared-frequencies than do the lowest of the modes kept (which we find are bounded below by $\omega^2 = 0$).

Although none of the backgrounds we perturb are supersymmetric, we find they are all marginally stable to perturbations which are well-behaved at the brane positions. The marginal direction is unique and is the one required on general grounds by a general scaling property of the 6D supergravity equations. Besides providing a general stability argument which works for a broad class of static vacua, we also provide analytic solutions to the fluctuation equation for the special case of conical backgrounds, including (but not restricted to) the rugby ball, (in comoving gauge) and the rugby ball solutions (in longitudinal gauge). These allow us to explicitly verify that the near-brane behaviour of the solutions has the properties required by general asymptotic properties of the field equations. By constructing the truncated action for the lowest-lying KK modes, we are able to show that no instabilities arise from mode-mixing amongst the lowest-lying levels, contrary to recent claims.

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A. Analysis in longitudinal gauge

In this section, we linearize the field equations in longitudinal gauge, specializing to those background configurations for which $\lambda_3 = 0$ and so which only have conical singularities. Working in this gauge allows us to make contact with earlier calculations, and so for ease of comparison in this section we adopt the notation of these earlier workers. Following refs. [26, 14] we therefore write

$$A^{(l)} = -\frac{\Psi}{2}\,, \quad V^{(l)} = \frac{\xi}{2}\,, \quad B^{(l)} = \Psi - \frac{\xi}{2}\,, \quad \Phi^{(l)} = -\frac{f}{2} \quad \text{and} \quad \mathcal{A}_\theta^{(l)} = \mathcal{A}_\theta\,. \tag{A.1}$$

We also follow refs. [26, 14] and adopt as radial coordinate, ρ, as proper distance in the extra dimensions (measured using the background metric), leading to the full ansatz

$$\mathrm{d}s^2 = \mathcal{W}^2(\rho)\, e^{-\Psi}\, \eta_{\mu\nu}\, \mathrm{d}x^\mu \mathrm{d}x^\nu + e^{\xi}\, \mathrm{d}\rho^2 + \mathcal{A}^2\, e^{2\Psi-\xi}\, \mathrm{d}\theta^2\,, \tag{A.2}$$

together with $e^{-2\varphi} = \mathcal{W}^4\, e^f$ and $A'_\theta = (q\mathcal{A}/\mathcal{W}^6) + \mathcal{A}'_\theta$. Throughout this appendix we use primes to denote $\mathrm{d}/\mathrm{d}\rho$ rather than $\mathrm{d}/\mathrm{d}\eta$.

Fourier transforming in time allows us to write $-\partial_t^2 = \omega^2$, and so we find that the (tt) and (ij) Einstein equations degenerate into the same equation at the linearized level, to give the following $(\mu\nu)$ Einstein equation:

$$\begin{aligned}&\frac{\omega^2\Psi}{\mathcal{W}^2} + \Psi'' + \left(\frac{6\,\mathcal{W}'}{\mathcal{W}} + \frac{\mathcal{A}'}{\mathcal{A}}\right)\Psi' + \left(\frac{2\,\mathcal{W}'}{\mathcal{W}}\right)\xi' \\ &\qquad -\frac{\kappa^2}{2}\left(\frac{q^2}{\mathcal{W}^{10}}\right)\left(2\Psi - \xi - \frac{f}{2}\right) + \kappa^2\left(\frac{q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}'_\theta - \frac{2\,g^2}{\kappa^2\mathcal{W}^2}\left(\xi - \frac{f}{2}\right) = 0\,.\end{aligned} \tag{A.3}$$

Similarly, the $(t\rho)$ equation becomes:

$$i\omega\left[\Psi' + \xi' + 2\left(\frac{\mathcal{W}'}{\mathcal{W}} - \frac{\mathcal{A}'}{\mathcal{A}}\right)\Psi + 2\left(\frac{\mathcal{W}'}{\mathcal{W}} + \frac{\mathcal{A}'}{\mathcal{A}}\right)\xi - \left(\frac{2\,\mathcal{W}'}{\mathcal{W}}\right)f - 2\kappa^2\left(\frac{q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}_\theta\right] = 0\,. \tag{A.4}$$

The $(\rho\rho)$ Einstein equation becomes

$$\begin{aligned}&-\frac{\omega^2\xi}{2\,\mathcal{W}^2} + \Psi'' + \frac{\xi''}{2} + \left(\frac{4\,\mathcal{W}'}{\mathcal{W}} - \frac{2\,\mathcal{A}'}{\mathcal{A}}\right)\Psi' + \frac{1}{2}\left(\frac{4\,\mathcal{W}'}{\mathcal{W}} + \frac{3\,\mathcal{A}'}{\mathcal{A}}\right)\xi' - \left(\frac{2\,\mathcal{W}'}{\mathcal{W}}\right)f' \\ &\qquad +\frac{3\kappa^2}{4}\left(\frac{q^2}{\mathcal{W}^{10}}\right)\left(2\Psi - \xi - \frac{f}{2}\right) - \frac{3\kappa^2}{2}\left(\frac{q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}'_\theta - \frac{g^2}{\kappa^2\mathcal{W}^2}\left(\xi - \frac{f}{2}\right) = 0\,.\end{aligned} \tag{A.5}$$

The $(\theta\theta)$ Einstein equation becomes

$$\begin{aligned}&-\frac{\omega^2\Psi}{\mathcal{W}^2} + \frac{\omega^2\xi}{2\,\mathcal{W}^2} - \Psi'' + \frac{\xi''}{2} - \left(\frac{4\,\mathcal{W}'}{\mathcal{W}}\right)\Psi' + \frac{1}{2}\left(\frac{4\,\mathcal{W}'}{\mathcal{W}} + \frac{3\,\mathcal{A}'}{\mathcal{A}}\right)\xi' \\ &\qquad +\frac{3\kappa^2}{4}\left(\frac{q^2}{\mathcal{W}^{10}}\right)\left(2\Psi - \xi - \frac{f}{2}\right) - \frac{3\kappa^2}{2}\left(\frac{q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}'_\theta - \frac{g^2}{\kappa^2\mathcal{W}^2}\left(\xi - \frac{f}{2}\right) = 0\,.\end{aligned} \tag{A.6}$$

The dilaton equation becomes

$$\frac{\omega^2 f}{\mathcal{W}^2} + f'' - \left(\frac{4\,\mathcal{W}'}{\mathcal{W}}\right)\left(\Psi' + \xi'\right) + \left(\frac{4\,\mathcal{W}'}{\mathcal{W}} + \frac{\mathcal{A}'}{\mathcal{A}}\right) f' \tag{A.7}$$
$$+\kappa^2\left(\frac{q^2}{\mathcal{W}^{10}}\right)\left(2\Psi - \xi - \frac{f}{2}\right) - 2\kappa^2\left(\frac{q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}_\theta' + \frac{4\,g^2}{\kappa^2\mathcal{W}^2}\left(\xi - \frac{f}{2}\right) = 0\,,$$

and the Maxwell equation becomes

$$\frac{\omega^2\mathcal{A}_\theta}{\mathcal{W}^2} + \mathcal{A}_\theta'' + \left(\frac{6\,\mathcal{W}'}{\mathcal{W}} - \frac{\mathcal{A}'}{\mathcal{A}}\right)\mathcal{A}_\theta' - \left(\frac{q\mathcal{A}}{\mathcal{W}^6}\right)\left(3\Psi' - \frac{f'}{2}\right) = 0\,. \tag{A.8}$$

Nominally we have here 6 equations — dilaton, Maxwell, and the (tt), $(t\rho)$, $(\rho\rho)$ and $(\theta\theta)$ Einstein equations — for the 4 unknown functions, Ψ, ξ, f and $\mathcal{A}_\theta$. However, as was the case in comoving gauge, the Bianchi identities ensure that two of these equations are not independent. This is because two combinations of these equations can be chosen to only depend on single derivatives with respect to time, and the Bianchi identities ensure that the solutions to these lower-order constraint equations is consistent with the time evolution. This allows us to use the constraint equations to solve for two of the fields, and then to drop two of the other field equations as redundant.

Following this logic we determine $\mathcal{A}_\theta$ by using the $(t\rho)$ Einstein equation, which integrates to

$$\left(\frac{\kappa^2 q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}_\theta = \frac{1}{2}(\Psi' + \xi') + \left(\frac{\mathcal{W}'}{\mathcal{W}} - \frac{\mathcal{A}'}{\mathcal{A}}\right)\Psi + \left(\frac{\mathcal{W}'}{\mathcal{W}} + \frac{\mathcal{A}'}{\mathcal{A}}\right)\xi - \left(\frac{\mathcal{W}'}{\mathcal{W}}\right) f + \Omega(\rho)\,, \tag{A.9}$$

where $\Omega(\rho)$ is an arbitrary function of integration. This unknown function can be determined by using the equation obtained by adding the $(tt) + (\rho\rho) + (\theta\theta)$ Einstein equations (which is a constraint inasmuch as it does not contain any time derivatives):

$$\Psi'' + \xi'' + \left(\frac{6\,\mathcal{W}'}{\mathcal{W}} - \frac{\mathcal{A}'}{\mathcal{A}}\right)\Psi' + \left(\frac{6\,\mathcal{W}'}{\mathcal{W}} + \frac{3\,\mathcal{A}'}{\mathcal{A}}\right)\xi' - \left(\frac{2\mathcal{W}'}{\mathcal{W}}\right) f' \tag{A.10}$$
$$+\left(\frac{\kappa^2 q^2}{\mathcal{W}^{10}}\right)\left(2\Psi - \xi - \frac{f}{2}\right) - \left(\frac{2\kappa^2 q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}_\theta' - \frac{4g^2}{\kappa^2\mathcal{W}^2}\left(\xi - \frac{f}{2}\right) = 0\,.$$

Substituting eq. (A.9) into eq. (A.10) and using the background equations of motion leads to the following condition for Ω:

$$(\mathcal{A}\mathcal{W}^4\Omega)' = 0\,, \tag{A.11}$$

and using this in the derivative of eq. (A.9) gives

$$\left(\frac{\kappa^2 q}{\mathcal{A}\mathcal{W}^4}\right)\mathcal{A}_\theta' = \frac{1}{2}(\Psi'' + \xi'') + \left(\frac{3\,\mathcal{W}'}{\mathcal{W}} - \frac{\mathcal{A}'}{2\,\mathcal{A}}\right)\Psi' + \left(\frac{3\,\mathcal{W}'}{\mathcal{W}} + \frac{3\,\mathcal{A}'}{2\,\mathcal{A}}\right)\xi' - \left(\frac{\mathcal{W}'}{\mathcal{W}}\right) f'$$
$$+\frac{1}{2}\left(\frac{\kappa^2 q^2}{\mathcal{W}^{10}}\right)\left(2\Psi - \xi - \frac{f}{2}\right) - \frac{2g^2}{\kappa^2\mathcal{W}^2}\left(\xi - \frac{f}{2}\right)\,. \tag{A.12}$$

We may now use this last expression to eliminate $\mathcal{A}'_\theta$ from the remaining 3 field equations, which we take to be the dilaton and (tt) and $(\rho\rho)$ Einstein equations, leading to:

$$\frac{\omega^2\Psi}{\mathcal{W}^2}+\frac{3}{2}\Psi''+\frac{1}{2}\,\xi''+\left(\frac{9\,\mathcal{W}'}{\mathcal{W}}+\frac{\mathcal{A}'}{2\,\mathcal{A}}\right)\Psi'+\left(\frac{5\,\mathcal{W}'}{\mathcal{W}}+\frac{3\,\mathcal{A}'}{2\,\mathcal{A}}\right)\xi'-\left(\frac{\mathcal{W}'}{\mathcal{W}}\right)f'-\frac{4\,g^2}{\kappa^2\mathcal{W}^2}\left(\xi-\frac{f}{2}\right)=0\,, \tag{A.13}$$

$$\frac{\omega^2\xi}{\mathcal{W}^2}-\frac{1}{2}\Psi''+\frac{1}{2}\,\xi''+\left(\frac{\mathcal{W}'}{\mathcal{W}}+\frac{5\,\mathcal{A}'}{2\,\mathcal{A}}\right)\Psi'+\left(\frac{5\,\mathcal{W}'}{\mathcal{W}}+\frac{3\,\mathcal{A}'}{2\,\mathcal{A}}\right)\xi'+\left(\frac{\mathcal{W}'}{\mathcal{W}}\right)f'-\frac{4\,g^2}{\kappa^2\mathcal{W}^2}\left(\xi-\frac{f}{2}\right)=0\,, \tag{A.14}$$

and

$$\begin{aligned}\frac{\omega^2 f}{\mathcal{W}^2}+f''-\Psi''-\xi''-\left(\frac{10\,\mathcal{W}'}{\mathcal{W}}-\frac{\mathcal{A}'}{\mathcal{A}}\right)\Psi'-\left(\frac{10\,\mathcal{W}'}{\mathcal{W}}+\frac{3\,\mathcal{A}'}{\mathcal{A}}\right)\xi'&\\ +\left(\frac{6\,\mathcal{W}'}{\mathcal{W}}+\frac{\mathcal{A}'}{\mathcal{A}}\right)f'+\frac{8\,g^2}{\kappa^2\mathcal{W}^2}\left(\xi-\frac{f}{2}\right)&=0\,,\end{aligned} \tag{A.15}$$

in agreement with eqs. (37)–(39) of ref. [14]. These equations easily lend themselves to numerical integration because they are independent, since all of the constraints have been explicitly solved.

The above equations take a somewhat simpler form if rewritten in terms of new variables: $\chi=2\Psi+f=-4A-2\Phi=2(B+V-\Phi)$ and $\Gamma=\Psi+\xi=2(V-A)=B+3V$:

$$\frac{\omega^2\chi}{\mathcal{W}^2}+\chi''+\left(\frac{4\,\mathcal{W}'}{\mathcal{W}}+\frac{\mathcal{A}'}{\mathcal{A}}\right)\chi'=0\,, \tag{A.16}$$

$$\frac{\omega^2\Gamma}{\mathcal{W}^2}+\Gamma''+\left(\frac{10\,\mathcal{W}'}{\mathcal{W}}+\frac{3\,\mathcal{A}'}{\mathcal{A}}\right)\Gamma'-\left(\frac{8\,g^2}{\kappa^2\mathcal{W}^2}\right)\Gamma=-\left(\frac{4\,g^2}{\kappa^2\mathcal{W}^2}\right)\chi\,, \tag{A.17}$$

and

$$\frac{\omega^2 f}{\mathcal{W}^2}+f''+\left(\frac{6\,\mathcal{W}'}{\mathcal{W}}-\frac{\mathcal{A}'}{\mathcal{A}}\right)f'=-\frac{\omega^2\Gamma}{\mathcal{W}^2}-\left(\frac{2\,\mathcal{A}'}{\mathcal{A}}\right)\chi'\,. \tag{A.18}$$

The logic to solving these is to integrate eq. (A.16) for χ, and then to use the result as a source in eqs. (A.17) and (A.18) to integrate these. These equations may be solved in closed form for the case of the rugby-ball solutions, as we show in what follows.

Although we arrive at equations which are in perfect agreement with ref. [14], we shall differ in the conclusion we draw from them inasmuch as we find the system to be marginally stable (modulo an issue concerning boundary conditions which is irrelevant for the comparison with [14]). For this reason we also review the stability analysis of ref. [14] in appendix B and explain the origin of the discrepancy.

A.1 Analytic Solutions for the Rugby Ball

The rugby ball is the special conical geometry for which $e^{w_0}=e^{a_0}=\mathcal{W}=e^{\varphi_0}=1$. In this case explicit forms for the axially-symmetric scalar mode functions can be found analytically in longitudinal gauge in terms of well-known special functions. This has the advantage of providing a concrete verification of the previous stability arguments, as well as providing a framework within which to see how the boundary conditions give rise to the quantization of Kaluza-Klein frequencies.

Recalling that in this section, we use the proper distance, ρ, as radial coordinate the rugby-ball geometry is given by

$$e^{b_0} = \mathcal{A} = \lambda\, r_0 \sin\left(\frac{\rho}{r_0}\right) \quad \text{where} \quad r_0 = \frac{\kappa}{2g}\,. \tag{A.19}$$

These choices imply $\lambda = 1 - \delta$ encodes the defect angle, $2\pi\delta$, at the branes situated at $\rho = 0$ and $\rho = \pi r_0$.

Using the rugby-ball conditions $\mathcal{W}' = 0$ and $\mathcal{A}'/\mathcal{A} = r_0^{-1}\cot(\rho/r_0)$ allows the fluctuation equations in longitudinal gauge, eqs. (A.16)–(A.18), to be simplified to

$$\begin{aligned} \omega^2\chi + \chi'' + \frac{\cot(\rho/r_0)}{r_0}\,\chi' &= 0 \\ \omega^2\Gamma + \Gamma'' + \frac{3\cot(\rho/r_0)}{r_0}\,\Gamma' - \frac{2\,\Gamma}{r_0^2} &= -\frac{\chi}{r_0^2} \\ \omega^2 f + f'' - \frac{\cot(\rho/r_0)}{r_0}\,f' &= -\omega^2\Gamma - \frac{2\cot(\rho/r_0)}{r_0}\,\chi'\,. \end{aligned} \tag{A.20}$$

To solve these equations we write $\omega^2 = \mu^2/r_0^2$ and change variables to $z = \cos(\rho/r_0)$, leading to a relatively familiar set of equations

$$\begin{aligned} \frac{\mathrm{d}^2\chi}{\mathrm{d}z^2} - \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\chi}{\mathrm{d}z} + \left(\frac{\mu^2}{1-z^2}\right)\chi &= 0 \\ \frac{\mathrm{d}^2\Gamma}{\mathrm{d}z^2} - \left(\frac{4z}{1-z^2}\right)\frac{\mathrm{d}\Gamma}{\mathrm{d}z} + \left(\frac{\mu^2-2}{1-z^2}\right)\Gamma &= -\frac{\chi}{1-z^2} \\ \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} + \left(\frac{\mu^2}{1-z^2}\right) f &= -\left(\frac{\mu^2}{1-z^2}\right)\Gamma + \left(\frac{2z}{1-z^2}\right)\frac{\mathrm{d}\chi}{\mathrm{d}z}\,. \end{aligned} \tag{A.21}$$

In terms of the variable $z = \cos(\rho/r_0)$ the brane singularities correspond to $z = \pm 1$, at which points we have

$$\begin{aligned} z &= 1 - \frac{\rho^2}{2} + \cdots\,, \quad \text{near } \rho = 0 \\ z &= -1 + \frac{(\rho - \pi r_0)^2}{2} + \cdots\,, \quad \text{near } \rho = \pi r_0\,, \end{aligned} \tag{A.22}$$

and so $\rho - \rho_\pm = \mp\sqrt{2}(1 \pm z)^{1/2} + \cdots$ where $\rho_+ = \pi r_0$ and $\rho_- = 0$. The asymptotic near-brane boundary conditions are therefore related to the behaviour of the solutions as $z \to \pm 1$. The asymptotic form which is appropriate to longitudinal gauge may be found by performing the transformation from GN gauge, along the lines as was done in the main text for comoving gauge. In this section we concentrate on finding those solutions for which all perturbations are less singular than $(1 - z^2)^{-1} \propto \rho^{-2}$, including in particular those for which the divergence at $z \to \pm 1$ is at most logarithmic.

Homogeneous solutions

We start by providing explicit integrals of the homogeneous parts of eqs. (A.21), for which the right-hand sides vanish. The first of these is the Legendre equation and therefore has as solutions

$$\chi(z) = C_1\, P_\nu(z) + C_2 \,\mathrm{Re}\, Q_\nu(z)\,, \tag{A.23}$$

where $\nu(\nu+1) = \mu^2$, or $\nu = \nu_\pm = \frac{1}{2}[-1 \pm (1+4\mu^2)^{1/2}]$. We take the real part of Q_ν here because this function is normally complex for $|z| < 1$. (See Appendix D for our detailed conventions.) Since $\nu_- = -1 - \nu_+$ and the differential equation is invariant under $\nu \to -1-\nu$ it suffices in what follows to use only the positive root, $\nu = \frac{1}{2}[\sqrt{1+4\mu^2} - 1]$.

For generic ν both P_ν and Q_ν have logarithmic singularities as $z \to \pm 1$. However, we shall find that for generic ν, Γ diverges like $(1 \pm z)^{-1}$. Restricting to at most logarithmic solutions therefore requires us to take $\nu = \ell = 0, 1, 2, \dots$, and so we only consider these values from this point on. (However, we emphasize that since logarithmic singularities are physically acceptable for the present problem, this quantization is *not* required by the boundary conditions for χ.) The mass formula

$$\omega^2 = \frac{\ell(\ell+1)}{r_0^2} = \frac{4g^2}{\kappa^2}\,\ell(\ell+1)\,, \tag{A.24}$$

then shows that stability directly follows from this quantization, in agreement with the general arguments of earlier sections.

Given $\nu = \ell$ the hypergeometric series for $P_\ell(z)$ terminates and degenerates to the Legendre polynomials, which are regular at *both* $z = \pm 1$. Nevertheless, since $Q_\nu(z)$ has the asymptotic limit

$$Q_\nu(z) = \pm \frac{1}{2\,(\pm 1)^\nu}\Big[-\ln\left(1 - z^{-2}\right) + O(1)\Big] \quad \text{as } z \to \pm 1\,, \tag{A.25}$$

it clearly retains its logarithmic behaviour at $z \to \pm 1$ even when $\nu = \ell$. Combining the above expressions allows the near-brane singularities to be written in terms of the remaining integration constants, C_1 and C_2, as follows:

$$\begin{aligned} \chi(z) &= -\frac{C_2}{2}\,\ln(1-z) + O(1) &\qquad \text{as } z \to 1 \\ &= \frac{C_2}{2(-1)^\ell}\,\ln(1+z) + O(1) &\qquad \text{as } z \to -1\,. \end{aligned} \tag{A.26}$$

The homogeneous part of the equation for Γ is also of Hypergeometric form, leading to the following general homogeneous solutions:

$$\Gamma_h(z) = C_3 F\left[1 + \frac{\nu}{2}, \frac{1}{2} - \frac{\nu}{2}; \frac{1}{2}; z^2\right] + C_4\, zF\left[\frac{3}{2} + \frac{\nu}{2}, 1 - \frac{\nu}{2}; \frac{3}{2}; z^2\right]\,. \tag{A.27}$$

Notice that the function multiplied by C_3 (or by C_4) is even (or odd) under reflections about the 'equator' at $\rho = \pi r_0/2$, since these correspond to $z \to -z$. In the special case $\nu = 0$ the

Hypergeometric series can be summed explicitly to give

$$F[1,b;b;z^2] = \sum_{k=0}^{\infty} z^{2k} = \frac{1}{1-z^2}\,, \tag{A.28}$$

showing that in this case the two homogeneous solutions are the elementary functions $(1-z^2)^{-1}$ and $z(1-z^2)^{-1}$.

The asymptotic behaviour of the Hypergeometric functions as $z \to \pm 1$ turn out to be

$$\begin{aligned}
F\left[1+\frac{\nu}{2},\frac{1}{2}-\frac{\nu}{2};\frac{1}{2};z^2\right] &= \frac{\sqrt{\pi}}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}\left[\frac{1}{1-z^2}-\frac{\nu}{2}\left(\frac{1}{2}+\frac{\nu}{2}\right)\ln(1-z^2)+O(1)\right]\,,\\
F\left[1-\frac{\nu}{2},\frac{3}{2}+\frac{\nu}{2};\frac{3}{2};z^2\right] &= \frac{\sqrt{\pi}}{2\,\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu}{2}\right)}\left[\frac{1}{1-z^2}-\frac{\nu}{2}\left(\frac{1}{2}+\frac{\nu}{2}\right)\ln(1-z^2)+O(1)\right]\,,
\end{aligned} \tag{A.29}$$

which shows in particular that Γ_h diverges like $(1-z^2)^{-1}$ for generic ν. Since we seek solutions which only diverge logarithmically, we must choose either C_3 or C_4 to vanish, and then choose ν so that the Hypergeometric series for the other solution must terminate. That is, either $C_4 = 0$ and $\frac{1}{2}-\frac{\nu}{2} = -n$ (and so $\nu = 1+2n$) or $C_3 = 0$ and $1-\frac{\nu}{2} = -n$ (and so $\nu = 2(n+1)$), where $n = 0,1,2,...$ is a non-negative integer. These are automatically satisfied for non-negative integers, $\nu = \ell \neq 0$, with even (odd) ℓ corresponding to $C_3 = 0$ ($C_4 = 0$). For $\nu = 0$ we have seen that the homogeneous solution is $(C_3 + C_4\, z)/(1-z^2)$, which is only nonsingular at both $z=1$ and $z=-1$ if $C_3 = C_4 = 0$. We see in this way why it is that ν must be quantized to be a non-negative integer.

The nonsingular homogeneous solutions for Γ_h for the first few choices for ℓ are:

$$\begin{aligned}
&\ell = 0 \qquad \text{implies} \quad \Gamma_h = 0\,;\\
&\ell = 1 \qquad \text{implies} \quad \Gamma_h = C_3 F\left[\frac{3}{2},0;\frac{1}{2};z^2\right] = C_3\,;\\
&\ell = 2 \qquad \text{implies} \quad \Gamma_h = C_4\, zF\left[\frac{5}{2},0;\frac{3}{2};z^2\right] = C_4\, z\,;\\
&\ell = 3 \qquad \text{implies} \quad \Gamma_h = C_3 F\left[\frac{5}{2},-1;\frac{1}{2};z^2\right] = C_3(1-5z^2)\,;
\end{aligned}$$

and so on.

The homogeneous part of the equation for f is also Hypergeometric, and is given in this case by

$$f_h(z) = C_5(1-z^2)F\left[1+\frac{\nu}{2},\frac{1}{2}-\frac{\nu}{2};\frac{1}{2};z^2\right] + C_6\, z(1-z^2)F\left[\frac{3}{2}+\frac{\nu}{2},1-\frac{\nu}{2};\frac{3}{2};z^2\right]\,. \tag{A.30}$$

Notice that these are the same Hypergeometric functions as appear in Γ, and so their singular properties can be read off from the expressions given above for Γ_h. In particular, because of the additional factor of $(1-z^2)$ the asymptotic forms do not diverge as $z \to \pm 1$ even if the Hypergeometric series does not terminate, leading to the limits

$$
\begin{aligned}
(1-z^2)F\left[1+\frac{\nu}{2},\frac{1}{2}-\frac{\nu}{2};\frac{1}{2};z^2\right] &\to \frac{\sqrt{\pi}}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}\,, \\
(1-z^2)F\left[1-\frac{\nu}{2},\frac{3}{2}+\frac{\nu}{2};\frac{3}{2};z^2\right] &\to \frac{\sqrt{\pi}}{2\,\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu}{2}\right)}\,.
\end{aligned}
\tag{A.31}
$$

For the first few choices for ℓ this leads to the following homogeneous solutions

$$
\begin{aligned}
\ell = 0 \quad \text{implies} \quad f_h &= C_5 + C_6\,z\,; \\
\ell = 1 \quad \text{implies} \quad f_h &= (1-z^2)\left(C_5 + C_6\,zF\left[2,\frac{1}{2};\frac{3}{2};z^2\right]\right)\,; \\
&= C_5(1-z^2) - \frac{C_6\,z}{4}\left[(1-z^2)\ln\left(\frac{1+z}{1-z}\right) - z^2\right] \\
\ell = 2 \quad \text{implies} \quad f_h &= (1-z^2)\left(C_5\,F\left[2,-\frac{1}{2};\frac{1}{2};z^2\right] + C_6\,z\right)\,; \\
&= C_5\left[\frac{3\,z}{4}(1-z^2)\ln\left(\frac{1+z}{1-z}\right) + \frac{3\,z^2}{2} - 1\right] + C_6 z(1-z^2) \\
\ell = 3 \quad \text{implies} \quad f_h &= (1-z^2)\left(C_5(1-5z^2) + C_6\,zF\left[3,-\frac{1}{2};\frac{3}{2};z^2\right]\right)\,;
\end{aligned}
$$

and so on. We see that because of the quantization, $\nu = \ell$, the only logarithmic singularities in the solutions are those appearing through $Q_\ell(z)$ in χ.

Perturbing tensions

Dividing the perturbations into those which are even or odd under the reflection $z \to -z$ allows an even more precise interpretation. Finite perturbations which are even under this reflection correspond to perturbations which change the tension of both boundary branes in the same way, and so remain within the class of rugby-ball solutions (for which both brane tensions must be equal). Conversely, regular perturbations which are odd under $z \to -z$ correspond to perturbations which change the two brane tensions oppositely, and so describe excursions into the more general class of warped, conical solutions [2, 19, 20]. Logarithmically diverging perturbations describe the transition from the conical to nonconical class of solutions.

Particular integrals

We now return to the problem of solving the differential equations for Γ and f, including now the right-hand sides of these equations. Since the general solution is found by adding the general solution to the homogeneous equation to any particular solution, it suffices here to identify particular integrals to these differential equations, which we denote by $\Gamma_{pi}(z)$ and $f_{pi}(z)$. The general solutions are then simply $\Gamma(z) = \Gamma_h(z) + \Gamma_{pi}(z)$ and $f(z) = f_h(z) + f_{pi}(z)$.

For instance, if we specialize to nonsingular solutions at both $z = \pm 1$ we may choose $C_2 = 0$ and so take χ to be given by the lowest few Legendre polynomials. This leads to the following expressions for $\Gamma(z) = \Gamma_h(z) + \Gamma_{pi}(z)$:

$$\begin{aligned}
&\ell = 0 \qquad \text{implies} \quad \Gamma = \Gamma_{pi}(z) = \frac{C_1}{2}\,; \\
&\ell = 1 \qquad \text{implies} \quad \Gamma = C_3 F\left[\frac{3}{4}, 0; \frac{1}{2}; z^2\right] + \Gamma_{pi}(z) = C_3 + \frac{C_1 z}{4}\,; \\
&\ell = 2 \qquad \text{implies} \quad \Gamma = C_4\, z F\left[\frac{5}{2}, 0; \frac{3}{2}; z^2\right] + \Gamma_{pi}(z) = C_4\, z - \frac{C_1 z^2}{4}\,;
\end{aligned} \tag{A.32}$$

and so on.

With these solutions in hand we next find the relevant particular integrals for the differential equation for f. Again restricting to nonsingular solutions, specializing to the first few Legendre polynomials, and summing $f = f_h + f_{pi}$ gives:

$$\begin{aligned}
&\ell = 0 \quad \text{implies} \quad f(z) = C_5 + C_6 z\,; \\
&\ell = 1 \quad \text{implies} \quad f(z) = (1 - z^2)\left(C_5 + C_6\, z F\left[2, \frac{1}{2}; \frac{3}{2}; z^2\right]\right) - C_3 + \frac{3\, C_1 z}{4}\,; \\
&\ell = 2 \quad \text{implies} \quad f(z) = (1 - z^2)\left(C_5\, F\left[2, -\frac{1}{2}; \frac{1}{2}; z^2\right] + C_6\, z\right) + \frac{5\, C_1}{8}\Big(3\, z^2 - 1\Big) - C_4\, z\,;
\end{aligned} \tag{A.33}$$

and so on for as many modes as are desired.

A similar process can be undergone in the more general case where $C_2 \neq 0$, and so $\chi = C_1\, P_\ell(z) + C_2 \operatorname{Re} Q_\ell(z)$. In this case χ diverges logarithmically in the near-brane limit, and the particular integrals are no longer obtainable in such a simple closed form. Nonetheless they may be obtained numerically in terms of integrals over the right-hand-side of the equations weighted by appropriate combinations of Hypergeometric functions.

A.2 Stability Analysis

Using the explicit previous solutions and particularly their asymptotic behaviour, we give here a general argument in favour of stability for a broad class of boundary conditions.

As performed in section 5.1, we argue for stability directly with the equations of motion and relate the sign of the energy eigenvalue, ω^2, to the boundary conditions which the fluctuations satisfy near the positions of the source branes.

Proceeding as before, we multiply eq. (A.16) by $\mathcal{A}\mathcal{W}^4 \chi^*$; sum the result with its complex conjugate; and integrate the answer over the extra dimensions, to get:

$$\begin{aligned}
\omega^2 \int_0^{\rho_1} \mathrm{d}\rho\, \mathcal{A}\mathcal{W}^2\, |\chi|^2 &= -\frac{1}{2} \int_0^{\rho_1} \mathrm{d}\rho\, \Big\{ \mathcal{A}\mathcal{W}^4 [\chi^* \chi'' + (\chi^*)'' \chi] + (\mathcal{A}\mathcal{W}^4)' (|\chi|^2)' \Big\} \\
&= -\left[\frac{1}{2} (\mathcal{A}\mathcal{W}^4)(|\chi|^2)'\right]_0^{\rho_1} + \int_0^{\rho_1} \mathrm{d}\rho\, \mathcal{A}\mathcal{W}^4 |\chi'|^2\,.
\end{aligned} \tag{A.34}$$

We denote here the brane positions by $\rho = 0$ and $\rho = \rho_1$. Since $\sqrt{-g} = \mathcal{A}\mathcal{W}^4 > 0$ we see that $\omega^2 \geq 0$ follows provided we can show that χ satisfies boundary conditions for which the combination $\mathcal{A}\mathcal{W}^4 \mathrm{d}(|\chi|^2)/\mathrm{d}\rho$ vanishes at the brane positions.

In general the appropriate boundary conditions in longitudinal gauge for χ have $\chi \to (H\rho)^{\Delta_l}$ in the near-brane limit, $\rho \to 0$, with Δ_l being a power which is determined in terms of the asymptotics of the background configuration, and $\Delta_l = 0$ could correspond to perturbations which remain finite at the branes as well as those which diverge only logarithmically at the brane positions. Using this asymptotic limit we have

$$(\mathcal{A}\mathcal{W}^4)\frac{\mathrm{d}(|\chi|^2)}{\mathrm{d}\rho} \propto \rho \, \frac{\mathrm{d}(|\chi^2|)}{\mathrm{d}\rho} \propto \rho^{2\Delta_l} \,, \tag{A.35}$$

as $\rho \to 0$. This shows that the integral on the left-hand side generically diverges due to its singularity near the branes, unless the combination $\rho \, \mathrm{d}(|\chi|^2)/\mathrm{d}\rho$ is well defined at $\rho = 0$. For the brane at $\rho = \rho_1$ the same argument may be repeated using the coordinate $\hat{\rho} = \rho_1 - \rho$, which increases as one moves away from $\hat{\rho} = 0$. Keeping in mind that $\mathrm{d}\hat{\rho} = -\mathrm{d}\rho$ but $\sqrt{-g} \propto +\hat{\rho}$ as $\hat{\rho} \to 0$, we see that

$$- \left[\mathcal{A}\mathcal{W}^4 \, \mathrm{d}(|\chi|^2)/\mathrm{d}\rho\right]_0^{\rho_1} = + \left[\hat{\rho} \, \mathrm{d}(|\chi|^2)/\mathrm{d}\hat{\rho}\right]_{\hat{\rho}=0} + \left[\rho \, \mathrm{d}(|\chi|^2)/\mathrm{d}\rho\right]_{\rho=0} \,, \tag{A.36}$$

and so in the special case that these limits are nonsingular at the brane position (such as is assumed when the brane is represented as a delta-function source, or if both the background and perturbed geometries have only conical singularities) we see that $\omega^2 \geq 0$ provided that $(|\chi|^2)' \geq 0$ in the near-brane limit. Furthermore, in this case $\omega = 0$ if and only if $\chi' = 0$.

Since the above argument assumes that the eigenmode of interest has $\chi \neq 0$, the only remaining step which is required to conclude that the system is marginally stable is to separately show that $\omega^2 \geq 0$ in the special case where $\chi = 0$. In this case we may apply the same reasoning to eq. (A.17), after multiplying through by $\mathcal{A}^3\mathcal{W}^{10}\Gamma^*$, to get

$$\omega^2 \int_0^{\rho_1} \mathrm{d}\rho \, \mathcal{A}^3\mathcal{W}^8|\Gamma|^2 = - \left[\frac{1}{2}(\mathcal{A}^3\mathcal{W}^{10})(|\Gamma|^2)'\right]_0^{\rho_1} + \int_0^{\rho_1} \mathrm{d}\rho \, \left[(\mathcal{A}^3\mathcal{W}^{10})|\Gamma'|^2 + \frac{8\,g^2}{\kappa^2}\,(\mathcal{A}^3\mathcal{W}^8)|\Gamma|^2\right] . \tag{A.37}$$

The boundary condition can be examined as above, with $\mathcal{A}^3\mathcal{W}^{10} = (\mathcal{A}\mathcal{W}^4)^3/\mathcal{W}^2 \propto \rho^{3-2\alpha_0}$. Since $\alpha_0 = 0$ for conical backgrounds this vanishes as ρ^3. We see again that $\omega^2 \geq 0$ if $\rho^3 \mathrm{d}(|\Gamma|^2)/\mathrm{d}\rho$ is well-defined and non-negative at the brane position, $\rho \to 0$. In this case $\omega = 0$ implies $\Gamma' = \Gamma = 0$.

This leaves open only the special case of modes for which both χ and Γ vanish, and for these modes we repeat the reasoning by multiplying eq. (A.18) through by $(\mathcal{W}^6/\mathcal{A})f^*$, leading to

$$\omega^2 \int_0^{\rho_1} \mathrm{d}\rho \, \left(\frac{\mathcal{W}^4}{\mathcal{A}}\right)|f|^2 = - \left[\frac{1}{2}\left(\frac{\mathcal{W}^6}{\mathcal{A}}\right)(|f|^2)'\right]_0^{\rho_1} + \int_0^{\rho_1} \mathrm{d}\rho \, \left(\frac{\mathcal{W}^6}{\mathcal{A}}\right)|f'|^2 . \tag{A.38}$$

In this case $\mathcal{W}^6/\mathcal{A} = (\mathcal{A}\mathcal{W}^4)(\mathcal{W}/\mathcal{A})^2 \propto \rho^{1+2(\alpha_0-\beta_0)} \propto \rho^{(3-5\beta_0)/2}$. Since $\beta_0 = 1$ for conical backgrounds this varies as ρ^{-1} as $\rho \to 0$, so if $\rho^{-1} \, \mathrm{d}(|f|^2)/\mathrm{d}\rho$ is well defined and non-negative at the brane position, $\rho \to 0$, we find $\omega^2 \geq 0$ with $\omega = 0$ if and only if $f' = 0$.

In this way we see how conclusions about the stability of the system can be related to asymptotic near-brane behaviour. Although the argument is inconclusive when the near-brane limit of the fluctuations is too singular, we do see that when the bulk fluctuations are nonsingular at the brane positions then the system is marginally stable. The stability is only marginal because of the known zero mode, for which $\Gamma = \chi = 0$ while f is a nonzero constant.

B. Comparison with previous work

The previous stability argument, together with the one performed in comoving gauge in section 5, seems in contradiction with the results of ref.[14]. We therefore briefly review their analysis in this section and pin down the reason of the discrepancy.

To analyze the system of linearized equations (A.18, A.17, A.16), ref. [14] starts from the assumption that all modes are proportional to the curvature perturbation Ψ (see eq. (41) of [14]). Using this assumption, they find that the only existing modes are of the form

$$\begin{pmatrix} f = 2\Psi \\ \Gamma = 2\Psi \\ \chi = 4\Psi \end{pmatrix}, \qquad \begin{pmatrix} f = -2\Psi \\ \Gamma = 2\Psi \\ \chi = 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\omega^2}{\mathcal{W}^2}\Psi + \Psi'' + \left(\frac{6\,\mathcal{W}'}{\mathcal{W}} - \frac{\mathcal{A}'}{\mathcal{A}}\right)\Psi' = 0 \\ \Gamma = 0 \\ \chi = 0 \end{pmatrix}, \tag{B.1}$$

from which they conclude that "the spectrum of the scalar excitations consists of a zero mode, a first excited state with constant wavefunction and a tower of additional excited states with non-constant wavefunctions". However, some other modes are present which have been discarded in this analysis. When plugging the relation (41) of [14] into the equations of motion, the authors implicitly assume that their function of proportionality cannot depend on the mode mass, which is an unnecessary restriction. They then derive an action (59) for their set of modes, but since their set of modes is not complete, the resulting action does not necessarily give the correct inner product for the modes.

Even still, one could still argue that if a ghost were present among this restricted set of modes, it should still be present when the full set is properly considered. To see its effect, we concentrate for now on the ghost found in [14]. Following the prescription of [14], the ghost mode seems to have its origin in the two mixing modes ψ_1 and $\tilde{\psi}_2$ of mass $m^2 = 16g^2$ which couple in the action as

$$\begin{aligned} \mathcal{L}_{\psi_1,\tilde{\psi}_2} = \quad & \mathcal{A}\,\psi_1\left(\Box - 16g^2\right)\psi_1 + \mathcal{C}\,\tilde{\psi}_2\left(\Box - 16g^2\right)\tilde{\psi}_2 \\ & + \mathcal{B}\,\tilde{\psi}_2\left(\Box - 16g^2\right)\psi_1 + \mathcal{B}\,\psi_1\left(\Box - 16g^2\right)\tilde{\psi}_2. \end{aligned} \tag{B.2}$$

Due to a typo, the suggested way to diagonalize this action (ansatz (63)) does not get rid of the cross-terms. Instead of eq.(63), the correct sign for the change of variables should have been

$$\phi_\pm = \frac{1}{\sqrt{2}}\left(\left(\mp d + \sqrt{1+d^2}\right)\psi_1 \pm \tilde{\psi}_2\right), \tag{B.3}$$

with $d = (\mathcal{C} - \mathcal{A})/2B$. This gives rise to an action where both ϕ_+ and ϕ_- come in with a positive kinetic term for any values of the parameters. To convince ourselves, we may perform the equivalent change of variables:

$$\psi_1 = \sqrt{\frac{C}{2}}\,(\phi_+ + \phi_-)\,, \quad \tilde{\psi}_2 = \sqrt{\frac{A}{2}}\,(\phi_+ - \phi_-)\,, \tag{B.4}$$

so that the action simplifies to

$$\mathcal{L}_{\psi_1,\tilde{\psi}_2} = \mathcal{K}_+\,\phi_+\left(\Box - 16g^2\right)\phi_+ + \mathcal{K}_-\,\phi_-\left(\Box - 16g^2\right)\phi_-, \tag{B.5}$$

with

$$\begin{aligned}\mathcal{K}_\pm &= \mathcal{A}\,\mathcal{C} \pm \sqrt{\mathcal{A}\,\mathcal{C}}\,\mathcal{B} \\ &= \sqrt{\frac{1}{12}(1+10x^2+x^4)}\left(\sqrt{(1+x^2)^2+8x^2} \pm (1+x^2)\right) > 0,\end{aligned}$$

where we wrote $x = 4g/q$ and used their expressions (60)-(62) for $\mathcal{A}, \mathcal{B}, \mathcal{C}$, so that both $\mathcal{K}_+$ and $\mathcal{K}_-$ are *positive for all values of the parameter space*. Thus we do not recover the same result as the one argued in [14]. The presence of a ghost in their analysis seems to be due to a sign error in the change of variable (B.3).

In section 5.2, we re-derive the action exactly for the entire set of modes in the theory, working in the more physical comoving gauge, and find the stable action which reproduces the field equations for the fluctuations.

C. Decoupled Variables in Different Gauges

In this appendix we relate the variables $(\tilde{\chi}, \tilde{\psi}, \tilde{f})$ — in terms of which the fluctuation equations decouple — to the fluctuation variables in comoving and longitudinal gauges.

Comoving gauge

In comoving gauge we have $2A^{(c)} = -\Psi$ and so using the gauge transformations (3.9), we have

$$\begin{aligned}\tilde{f} &= \frac{\kappa q}{\mathcal{U}}\,e^{\mathcal{X}}\left(B^{(c)} + \frac{1}{2}\Phi^{(c)}\right) \\ \tilde{\chi} &= -\frac{\sqrt{3}}{4\mathcal{U}\mathcal{V}}\left[(2\mathcal{U}^2 + \mathcal{X}'\mathcal{Z}')\Phi^{(c)} - 2\mathcal{U}^2\Psi^{(c)} + 2\mathcal{X}'\mathcal{Z}'B^{(c)}\right] \\ \tilde{\psi} &= \left(\frac{2\mathcal{X}' - 3\mathcal{Z}'}{4\mathcal{V}}\right)\Phi^{(c)} + \left[\frac{3\mathcal{V}}{(2\mathcal{Y}' + \mathcal{Z}')} + \frac{3\mathcal{Z}'}{4\mathcal{V}}\right]\Psi^{(c)} - \left[\frac{2\mathcal{V}}{(2\mathcal{Y}' + \mathcal{Z}')} - \frac{\mathcal{X}'}{\mathcal{V}}\right]B^{(c)}\,,\end{aligned} \tag{C.1}$$

where primes denote differentiation with respect to η.

Longitudinal gauge

In longitudinal gauge our notation is, $\Phi^{(l)} = -f/2$, and $B^{(l)} = \Psi^{(l)} - \xi^{(l)}/2$. Using the gauge transformation (3.9) then leads to

$$
\begin{aligned}
\tilde{f} &= \frac{\kappa q}{\mathcal{U}}\, e^{\mathcal{X}} \left[\left(B^{(l)} + \frac{1}{2}\Phi^{(l)} \right) - \left(\frac{\mathcal{A}'}{\mathcal{A}} + \frac{1}{2}\varphi_0' \right) \frac{\mathcal{A}_\theta^{(l)}}{a_\theta'} \right] \\
\tilde{\chi} &= -\frac{\sqrt{3}}{4\mathcal{U}\mathcal{V}} \Bigg[(2\mathcal{U}^2 + \mathcal{X}'\mathcal{Z}')\Phi^{(l)} - 2\mathcal{U}^2\Psi^{(l)} + 2\mathcal{X}'\mathcal{Z}'B^{(l)} \\
&\quad - \left(2\mathcal{U}^2 \left[\varphi_0' + \frac{2\mathcal{W}'}{\mathcal{W}} \right] + \mathcal{X}'\mathcal{Z}' \left[\frac{2\mathcal{A}'}{\mathcal{A}} + \varphi_0' \right] \right) \frac{\mathcal{A}_\theta^{(l)}}{a_\theta'} \Bigg] \\
\tilde{\psi} &= \left(\frac{2\mathcal{X}' - 3\mathcal{Z}'}{4\mathcal{V}} \right) \left(\Phi^{(l)} - \varphi_0' \frac{\mathcal{A}_\theta^{(l)}}{a_\theta'} \right) + \left(\frac{3\mathcal{V}}{(2\mathcal{Y}' + \mathcal{Z}')} + \frac{3\mathcal{Z}'}{4\mathcal{V}} \right) \left(\Psi^{(l)} + \frac{2\mathcal{W}'}{\mathcal{W}} \frac{\mathcal{A}_\theta^{(l)}}{a_\theta'} \right) \\
&\quad - \left(\frac{2\mathcal{V}}{(2\mathcal{Y}' + \mathcal{Z}')} - \frac{\mathcal{X}'}{\mathcal{V}} \right) \left(B^{(l)} - \frac{\mathcal{A}'}{\mathcal{A}} \frac{\mathcal{A}_\theta^{(l)}}{a_\theta'} \right),
\end{aligned}
$$

where $\mathcal{A}_\theta^{(l)}$ should be related to the variables $(B_{(l)}, \Phi_{(l)}, \Psi_{(l)})$, (or $(\chi_{(l)}, \Gamma_{(l)}, f_{(l)})$) using the perturbed equation (A.9). We note that eq.(A.9) gives an expression for $\mathcal{A}_\theta^{(l)}$ up to an integration constant $\Omega(\rho)$. However Ω only contributes to the massless part of $\mathcal{A}_\theta^{(l)}$, for which it can be interpreted as a redefinition of the background quantities. We can therefore set Ω to zero in what follows.

For ease of reference we also give here the specialization of the above relations for the case of conical singularities, for which we have:

$$
\begin{aligned}
\tilde{\chi} &= \frac{\sqrt{3}}{4}\, \chi_{(l)} \\
\tilde{f} &= e^{-\mathcal{X}} \Bigg[-\frac{\mathcal{U}}{\kappa q}\, f^{(l)} + \left(\frac{3\mathcal{U}}{4\kappa q} + \frac{\mathcal{X}'\mathcal{Y}'}{4\kappa q\mathcal{U}} \right) \chi^{(l)} \\
&\quad - \left(\frac{\mathcal{U}}{2\kappa q} + \frac{\mathcal{X}'\mathcal{Y}'}{2\kappa q\mathcal{U}} \right) \Gamma^{(l)} - \frac{\mathcal{X}'}{2\kappa q\mathcal{U}}\, \Gamma^{(l)\prime} \Bigg] \\
\tilde{\psi} &= -\frac{\mathcal{Y}'}{4\mathcal{U}}\, \chi^{(l)} + \frac{1}{\mathcal{U}\mathcal{Y}'} \left(-\frac{2\, g^2}{\kappa^2}\, e^{2\mathcal{Y}} + \mathcal{U}^2 \right) \Gamma^{(l)} + \frac{1}{2\mathcal{U}} \Gamma^{(l)\prime}\,.
\end{aligned}
\tag{C.2}
$$

For consistency, one can check that the field equations (3.24, 3.25, 3.26) in the conical case together with this change of variable give back the equations in longitudinal gauge (A.16, A.17, A.18).

D. Some Special Functions

In this Appendix we record some useful properties of the special functions which are used in the main text. In our conventions the Legendre functions are related to Hypergeometric functions by

$$
\begin{aligned}
P_\nu(z) &= F\left[-\nu, 1+\nu; 1; \frac{1}{2}(1-z)\right] \\
Q_\nu(z) &= \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{(2z)^{1+\nu}\Gamma\left(\frac{3}{2}+\nu\right)}\, F\left[1+\frac{\nu}{2}, \frac{1}{2}+\frac{\nu}{2}; \frac{3}{2}+\nu; z^{-2}\right], \quad \nu \neq -\frac{3}{2}, -\frac{5}{2}, \dots
\end{aligned}
$$

The asymptotic form of the Legendre functions as $z \to \pm 1$ can be found from well-established properties of the hypergeometric functions, most notably its series definition

$$
F[a,b;c;z] = 1 + \frac{ab}{c}\, z + \frac{a(a+1)b(b+1)}{c(c+1)}\, \frac{z^2}{2!} + \cdots, \tag{D.1}
$$

provided $c \neq 0, -1, -2, \dots$. The potential singularities of $F[a,b;c;z]$ lie at $z = \pm 1$ and $z = \infty$, and our interest is in particular its behaviour at $z = \pm 1$. A useful identity for identifying these behaviours is

$$
\begin{aligned}
F[a,b;c;u] &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-b)\Gamma(c-a)}\, F[a,b;1+a+b-c;1-u] \\
&\quad + (1-u)^{c-a-b}\left[\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\right] F[c-a, c-b; c-a-b+1; 1-u]\,.
\end{aligned} \tag{D.2}
$$

This expression shows that $F[a,b;c;z]$ is finite as $z \to 1$ provided $c > a+b$, it diverges there logarithmically if $c = a+b$, and it can be more singular if $c < a+b$:

$$
\begin{aligned}
F[a,b;c;1] &= \frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)}, \qquad \text{if} \quad c > a+b,\; a,b \notin \mathbb{Z}_- \\
F[a,b;a+b;z] &\xrightarrow[z \to 1]{} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \log(1-z), \quad \text{if} \quad a,b \notin \mathbb{Z}_- \\
F[-n,b;c;1] &= \frac{(c-b)_n}{(c)_n} \quad n \in \mathbb{N}\,.
\end{aligned}
$$

Here Γ is Euler's Gamma function $\Gamma(n) = (n-1)!$, defined for complex arguments by $\Gamma(x) = \int_0^\infty \mathrm{d}u\; u^{x-1}e^{-u}$. $(\cdots)_n$ is the Pochhammer symbol, defined by $(x)_n = \frac{\Gamma(x+n)}{\Gamma(x)}$.

For instance, using eq. (D.2) with $a = -\nu$, $b = 1+\nu$, $c = 1$ and $u = \frac{1}{2}(1-z)$ leads (provided $\nu \neq 0, 1, 2, \dots$) to the following asymptotic expression

$$
\begin{aligned}
P_\nu(z) &= -\frac{\ln\left[\frac{1}{2}(1+z)\right]}{\Gamma(-\nu)\Gamma(1+\nu)}\, F\left[-\nu, 1+\nu\,; 1\,; \frac{1}{2}(1+z)\right] + O(1) \\
&= -\frac{\ln\left[\frac{1}{2}(1+z)\right]}{\Gamma(-\nu)\Gamma(1+\nu)} + O(1) \quad \text{as } z \to -1\,,
\end{aligned} \tag{D.3}
$$

If, on the other hand, $\nu = \ell = 0, 1, \dots$, then the hypergeometric series terminates and $P_\ell(z)$ is bounded at *both* $z = \pm 1$. This leads in the usual way to the Legendre polynomials, for which with $P_0(z) = 1$ and:

$$
P_\ell(z) = \frac{1}{2^\ell \ell!}\left(\frac{\mathrm{d}}{\mathrm{d}z}\right)^\ell \left(z^2 - 1\right)^\ell, \quad \ell = 1, 2, \dots\,, \tag{D.4}
$$

so that $P_1(z) = z$, $P_2(z) = \frac{1}{2}(3z^2-1)$, $P_3(z) = \frac{1}{2}\,z(5z^2-3)$, and so on.

Similarly, using $a = 1+\frac{\nu}{2}$, $b = \frac{1}{2}+\frac{\nu}{2}$, $c = \frac{3}{2}+\nu$ and $u = z^{-2}$ in the identity (D.2), leads to

$$
\begin{aligned}
Q_\nu(z) &= \frac{\sqrt{\pi}\,\Gamma(1+\nu)}{(2z)^{1+\nu}\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}+\frac{\nu}{2}\right)}\Big[-\ln\left(1-z^{-2}\right)\Big]\,F\left[1+\frac{\nu}{2},\frac{1}{2}+\frac{\nu}{2}\,;1\,;1-z^{-2}\right]+O(1)\\
&= \pm\frac{1}{2(\pm1)^\nu}\Big[-\ln\left(1-z^{-2}\right)+O\left(1\right)\Big] \quad \text{as } z\to\pm1\,,
\end{aligned}
\tag{D.5}
$$

where we have used properties of the Gamma function to provide further simplifications. This function clearly has logarithmic singularities at both $z = 1$ and $z = -1$. Notice that in the region of interest, $-1 < z < 1$, we have $1 - z^{-2} < 0$, leading to a function which is complex. Since χ is given by the real part of Q_ν we have the following expressions for the near-brane singularities of χ:

$$
\begin{aligned}
\chi(z) &= -\frac{C_2}{2}\ln\left(1-z\right)+O(1) \quad \text{as } z\to1\\
&= -\left\{\frac{C_1}{\Gamma(-\nu)\Gamma(1+\nu)}-\frac{C_2}{2(-1)^\nu}\right\}\ln(1+z)+O(1) \quad \text{as } z\to-1\,.
\end{aligned}
\tag{D.6}
$$

The homogeneous part of the equation for Γ is also of Hypergeometric form, leading to the following general homogeneous solutions:

$$
\Gamma_h(z) = C_3F\left[1+\frac{\nu}{2}\,,\frac{1}{2}-\frac{\nu}{2}\,;\frac{1}{2};z^2\right]+C_4\,zF\left[\frac{3}{2}+\frac{\nu}{2}\,,1-\frac{\nu}{2}\,;\frac{3}{2};z^2\right]. \tag{D.7}
$$

Notice that $F(a,b;b;x) = F(b,a;b;x) = 1+ax+\frac{1}{2}\,a(a+1)x^2+\cdots$ is independent of b, and so the two Hypergeometric functions in the above expression become identical to one another in the special case $\nu = 0$. In fact, for $\nu = 0$ the Hypergeometric series can be summed explicitly to give

$$
F[1,b;b;z^2] = \sum_{k=0}^\infty z^{2k} = \frac{1}{1-z^2}\,, \tag{D.8}
$$

showing that in this case the two homogeneous solutions are $(1-z^2)^{-1}$ and $z(1-z^2)^{-1}$.

The asymptotic behaviour of the Hypergeometric functions as $z\to\pm1$ is found by specializing eq. (D.2) to the cases (i) $a = 1+\frac{\nu}{2}$, $b = \frac{1}{2}-\frac{\nu}{2}$, $c = \frac{1}{2}$ and $u = z^2$; and (ii) $a = \frac{3}{2}+\frac{\nu}{2}$, $b = 1-\frac{\nu}{2}$, $c = \frac{3}{2}$ and $u = z^2$. Keeping in mind that $F[a,b\,;\epsilon\,;x] = 1+ab\,x/\epsilon+O(x^2)$, this leads to

$$
\begin{aligned}
F\left[1+\frac{\nu}{2}\,,\frac{1}{2}-\frac{\nu}{2}\,;\frac{1}{2}\,;z^2\right] &= \frac{\sqrt{\pi}}{\Gamma\left(1+\frac{\nu}{2}\right)\Gamma\left(\frac{1}{2}-\frac{\nu}{2}\right)}\left[\frac{1}{1-z^2}-\frac{\nu}{2}\left(\frac{1}{2}+\frac{\nu}{2}\right)\ln(1-z^2)+O(1)\right],\\
F\left[1-\frac{\nu}{2}\,,\frac{3}{2}+\frac{\nu}{2}\,;\frac{3}{2}\,;z^2\right] &= \frac{\sqrt{\pi}}{2\,\Gamma\left(1-\frac{\nu}{2}\right)\Gamma\left(\frac{3}{2}+\frac{\nu}{2}\right)}\left[\frac{1}{1-z^2}-\frac{\nu}{2}\left(\frac{1}{2}+\frac{\nu}{2}\right)\ln(1-z^2)+O(1)\right].
\end{aligned}
\tag{D.9}
$$

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